

# Conformal Field Theory of the Integer Quantum Hall Plateau Transition

Martin R. Zirnbauer\*

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## Abstract

A solution to the long-standing problem of identifying the conformal field theory governing the transition between quantized Hall plateaus of a disordered noninteracting 2d electron gas, is proposed. The theory is a nonlinear sigma model with a Wess-Zumino-Novikov-Witten term, and fields taking values in a Riemannian symmetric superspace based on  $H^3 \times S^3$ . Essentially the same conformal field theory appeared in very recent work on string propagation in  $AdS_3$  backgrounds. We explain how the proposed theory manages to obey a number of tight constraints, two of which are constancy of the partition function and noncriticality of the local density of states. An unexpected feature is the existence of a truly marginal deformation, restricting the extent to which universality can hold in critical quantum Hall systems. The marginal coupling is fixed by matching the short-distance singularity of the conductance between two interior contacts to the classical conductivity  $\sigma_{xx} = 1/2$  of the Chalker-Coddington network model. For this value, perturbation theory predicts a critical exponent  $X_t = 2/\pi$  for the typical point-contact conductance, in agreement with numerical simulations. The irrational exponent is tolerated by the fact that the symmetry algebra of the field theory is Virasoro but not Kac-Moody.

## 1 Introduction

A two-dimensional electron gas subjected to strong magnetic fields exhibits the striking phenomenon of the quantum Hall effect [1, 2]: as the temperature is lowered into the (sub-)Kelvin range, the Hall conductance evolves towards a

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\*Current address: Department of Mathematics, Fine Hall, Princeton University, USA.  
Permanent address: Institut für Theoretische Physik, Universität zu Köln, Zùlpicher Str. 77, 50937 Köln, Germany. Email: [zirn@thp.Uni-Koeln.DE](mailto:zirn@thp.Uni-Koeln.DE)

staircase function, with quantized plateau values  $\nu e^2/h$  occurring around certain magic Landau level filling fractions  $\nu$ . It is fair to say that the physical reasons for the quantization and its stability with respect to changing the magnetic field, are by now well understood [3]. What has proved more difficult to clarify is the precise nature of the *transitions between plateaus*. For all we know, these are associated with quantum critical points of the electron gas. Extrapolation to zero temperature and infinite system size indicates a second-order phase transition with a divergent correlation length. In the integer version of the effect, which we will be concerned with in the present paper, interactions between the electrons are thought to be irrelevant, and the critical behavior is attributed to the interplay between the disorder, which tends to localize the electrons, and the kinetic energy quenched by the strong magnetic field, causing delocalization for some isolated values of the Fermi energy [4]. The major theoretical goal is to identify the proper *low-energy effective field theory*. On general grounds, this theory is expected to be conformal at the transition, and should provide a systematic framework for the calculation of critical exponents and the formulation of a complete scaling theory [5, 6]. Needless to say, in spite of fifteen years of massive effort, that ambitious theoretical goal was never achieved.

The early development of the subject was guided by a field theory of the type of a  $G/H$  nonlinear sigma model, called Pruisken's model [7]. Its original formulation relied on a fermionic version of the replica trick, leading to a symmetry group  $G = \text{U}(2n)$  (with  $n = 0$ ) gauged on the right by  $H = \text{U}(n) \times \text{U}(n)$ . A mathematically satisfactory formulation avoiding replicas was introduced by Weidenmüller [8], who employed the supersymmetric method of Efetov [9]. In that variant,  $G$  is a pseudo-unitary supergroup  $\text{U}(1|1|2)$ , and  $H = \text{U}(1|1) \times \text{U}(1|1)$ . In either case, the field theory has two dimensionless coupling constants, denoted by  $\sigma_{xx}$  and  $\sigma_{xy}$ , which are identified with the longitudinal and Hall conductivities of the disordered electron gas. On the basis of Pruisken's model, Khmelnitskii [10] conjectured a renormalization group flow diagram for  $\sigma_{xx}$  and  $\sigma_{xy}$ , where the central feature is a fixed point of the flow at some  $\sigma_{xx}^*$  and  $\sigma_{xy}^* = 1/2$ . The fixed point was argued to have one relevant ( $\sigma_{xy} - 1/2$ ) and one irrelevant ( $\sigma_{xx} - \sigma_{xx}^*$ ) perturbation, giving a fair representation of experimental and numerical data on the transition. Unfortunately, the initial successes and guesses inspired by Pruisken's nonlinear sigma model were not backed up by a more complete solution. The problem is that the postulated fixed point — if it exists as a fixed point of Pruisken's model, which is not really clear — lies at strong coupling, or small  $\sigma_{xx}$ , where it cannot be controlled by its representation through  $G/H$  fields. (The one-instanton computations of Pruisken and collaborators [11] can only be trusted at weak coupling, and to learn something about the fixed point, a far extrapolation to strong coupling is required.) Consequently, no quantitative results beyond the rough, though instructive, picture of two-parameter scaling have ever come out of that theory.

The disillusionment about Pruisken's model grew into a forceful complaint [12] when a detailed understanding of a related 2d fixed point became available, namely that governing the six-vertex model with isotropic vertex weights or, equivalently, the sine-Gordon model at  $\beta^2 = 8\pi$ , or, the  $xy$ -model at the Kosterlitz-Thouless transition. Another of its many incarnations is found in isotropic 1d quantum antiferromagnets with half-integral spin, which following Haldane [13] map on the  $O(3)$  nonlinear sigma model with a topological term and topological angle  $\theta = \pi$ . In that paradigmatic system, it soon became clear [14] that the “good” theory, where conformal invariance is manifest and critical properties can be computed in great detail, is not the  $O(3)$  model but a Wess-Zumino-Novikov-Witten (WZW) model. Characteristically, the fields of the latter take values in a group (which is  $SU(2)$  here), whereas the target space of the  $O(3)$  model is a coset space  $SU(2)/U(1)$ . Thus the field is promoted from being coset-valued to group-valued, at criticality. Although antiferromagnets seem to be a far cry from disordered electrons, the lesson learned from this example does have a bearing on the quantum Hall critical point and ought to be taken seriously: Pruisken's replicated model at replica number  $n = 1$  (and critical  $\sigma_{xy} = 1/2$ ) is nothing else than the  $O(3)$  model at  $\theta = \pi$ , which in turn is also a basic constituent of the supersymmetric  $G/H$  model.

Drawing on the insight gained from antiferromagnets, a natural idea for making progress with the quantum Hall plateau transition is to try and promote the field space  $G/H$  to a group or group-like manifold. (Note that we are well advised to avoid the mathematical ambiguities [15, 16] of the replica trick and work with the supersymmetric formulation. Affleck attempted to execute the program in the replicated version, but failed [17].) Although this idea has now been around for more than a decade, progress was painfully slow. The reason was that there existed a number of severe obstacles, a chronological personal account of which is as follows.

Given the symmetries of the supersymmetric version of Pruisken's model, one might take a WZW model with target  $U(1, 1|2)$ , or  $SU(1, 1|2)$ , or  $PSU(1, 1|2)$ , for a first candidate. Here we stall immediately. In trying to solve the statistical physics problem at hand, we have to be very discriminating about which functional integral to accept as well-defined and which not. In concrete terms, we are looking for a field theory defined over *Euclidian* two-space, and with a target space of *Euclidian* signature. This constraint eliminates candidate theories with an action functional that is bounded neither from below nor from above. Among these are the above supergroups, the natural supergeometry of which is non-Riemann, or of indefinite signature. (The natural geometry is forced on us by symmetry considerations.)

Let us mention in passing that, to have a WZW model with definite metric, one option is to start from  $G = U(1, 1|2)$  and gauge by  $H = U(1|1) \times U(1|1)$ . What we have in mind here is the functional integral version [18] of the Goddard-

Kent-Olive construction. Unfortunately, in that construction the gauge group acts by *conjugation* (so it acts simultaneously on the left and right), which ruins conservation of some of the  $G$  currents. Symmetries are not lost under renormalization, and because  $G$  is the symmetry group of (the supersymmetric version of) Pruisken’s model, it *must* be present in the fixed-point theory. This kills the idea of gauging the  $G$ -WZW model by  $H$ .

The next attempt is to modify  $U(1, 1|2)$ , so as to arrive at a target manifold with better metric properties, giving an action functional bounded from below by the constant fields. Curiously, it turns out that the proper modification does not exist within the realm of standard supermanifold theory, but requires the introduction of objects I call *Riemannian symmetric superspaces* [19]. They belong to the general category of cs-manifolds [20]. The definition of these nonstandard notions, and their illustration at a well-chosen simple example, will be given in Section 6. The crucial feature of the “good” variant of  $U(1, 1|2)$  — let’s name it  $\mathbf{X}$  for short — is that it is based on a symmetric space  $M_B \times M_F$ ,  $M_B = GL(2, \mathbb{C})/U(2)$  and  $M_F = U(2)$ , which has the desired property of being *Riemann* in the geometry inherited from the natural supergeometry of  $\mathbf{X}$ . In some sense,  $U(1, 1|2)$  is one “real” form of the complex supergroup  $GL(2|2)$ , and  $\mathbf{X}$  is another. (In the notation of Section 6,  $\mathbf{X}$  is the symmetric superspace  $(GL(2|2), \text{STr}(g^{-1}dg)^2, M_B \times M_F)$ .) While  $\mathbf{X}$  is *not* a group, it does give rise to an acceptable and well-defined functional integral, the  $\mathbf{X}$ -WZW model. (By an abuse of terminology, one might also call it the  $GL(2|2)$  WZW model [21].)

Aside from its distinguished mathematical role, the  $\mathbf{X}$ -WZW model also has a noteworthy *physical* origin. Several models have been proposed as a starting point for the description of the quantum Hall plateau transition, and among them there is a model [22] of two-dimensional Dirac fermions subject to various types of disorder: random vector potential, random scalar potential, and random mass. For the purpose of computing the disorder-averaged Green functions and other quenched correlation functions of the model, one adds a bosonic  $\beta$ - $\gamma$  ghost system, which normalizes the partition function to unity [23]. By a superextension of Witten’s nonabelian bosonization scheme [24], the weakly disordered Dirac- $\beta\gamma$  system then transforms into a weakly perturbed WZW model. Actually, there exist two schemes [25], and the better one to use in the present context leads to the target being  $GL(1|1)$  for a single Green function, and  $GL(2|2)$  for two Green functions. These identifications of the target space are naive, and a careful analysis of the second case shows that proper use of the bosonization scheme leads to the target  $\mathbf{X}$ . Thus the pure Dirac- $\beta\gamma$  system is equivalent to the  $\mathbf{X}$ -WZW model (at level  $k = 1$ ), and the disorder which is present translates into perturbations of it.

The usefulness of such an approach now hinges on the nature of the perturbations. From [22] it is known that the random scalar potential, the random vector potential and the random mass are marginally relevant, truly marginal,

and marginally irrelevant, in that order. The dangerous perturbation by a random scalar potential, which is generic to the quantum Hall (QH) universality class, grows under renormalization and drives the Dirac- $\beta\gamma$  system to an unknown fixed point at strong coupling. Unfortunately, nonabelian bosonization does not give much of a clue as to how to handle that strong-coupling problem. Presumably, the fixed point can somehow be described in terms of fields taking values inside  $\mathbf{X}$ , but exactly what happens remains mysterious. (The situation is less favorable here than in Affleck’s trick [25] for passing from the critical Heisenberg chain via Dirac fermions to the  $SU(2)_1$  WZW model. In that case, the relevant perturbation simply reduces  $U(2)$  to  $SU(2)$ .) Thus we are stuck once again.

The next piece of confusing evidence came from a 1996 numerical study [26] of point-contact conductances in the Chalker-Coddington network model of the QH plateau transition. From the perspective of conformal field theory, the point-contact conductance is the most basic and “clean” observable, as Pruisken’s model expresses it as a *two-point function of local fields*. In a WZW model with Kac-Moody symmetry, the scaling dimensions of local (primary) fields  $\phi_\lambda$  transforming according to a representation  $\lambda$ , are given [27] by the formula  $\mathcal{C}_\lambda/(k+h_*)$ , where  $\mathcal{C}_\lambda$  is the quadratic Casimir invariant evaluated on  $\lambda$ , and the level  $k$  and the dual Coxeter number  $h_*$  are integers. In view of this fact, it has been termed striking that the critical exponent  $X_t$  for the typical point-contact conductance was found to be  $X_t = 0.640 \pm 0.009$ , which is numerically close to  $2/\pi \approx 0.637$ . The latter value, if exact, is hard to reconcile with the above formula, which predicts rational numbers. It seemed, at that point, that the idea of promoting the field space  $G/H$  to a group (or group-like manifold), and passing from Pruisken’s model to a WZW model, fails to work.

The present paper was triggered by the recent appearance of two articles [28, 29] related to superstring propagation on  $AdS_3$  backgrounds. There, a prominent role is played by a nonlinear sigma model with target  $PSU(1,1|2)$ , the supergroup obtained from  $U(1,1|2)$  by requiring unit superdeterminant and gauging w.r.t. the multiples of the unit matrix. The intriguing message from those articles is that the  $PSU(1,1|2)$  nonlinear sigma model is *conformal at any value of its coupling,  $f$* . The model also allows for the presence of a Wess-Zumino term with topological coupling  $k$ , so we have a two-parameter family of conformal field theories at our disposal. (In the string-theory context, the two coupling constants are related to the Ramond-Ramond and Neveu-Schwarz fluxes that are due to a number of fivebranes wrapped around some Calabi-Yau manifold.) The marginality of the coupling  $f$  looks very promising from our perspective, as it suggests enough flexibility to accommodate the peculiar critical exponent found for the typical point-contact conductance. We are thus led to reconsider the WZW model idea.

Our story does not converge to a quick conclusion, as there still exist a num-

ber of difficulties to overcome. First of all,  $\text{PSU}(1,1|2)$  is one of those target spaces we discarded right at the outset, on the grounds that we insist on having a target metric with Euclidian signature. Second, the Virasoro central charge of the  $\text{PSU}(1,1|2)$  nonlinear sigma model (with and without Wess-Zumino term) has the value  $c = -2$ , which is at variance with a basic constraint on the theory: its partition function must be identically equal to unity, independent of all parameters of the noninteracting electron gas. Third, correlation functions that involve only retarded or only advanced Green functions of the disordered electron system, are known to be noncritical (or even trivial). It is not a priori obvious how one can arrange for the  $\text{PSU}(1,1|2)$  model to reproduce this feature. Fourth, the marginality of the coupling  $f$ , while needed to accommodate the exponent  $X_t$ , appears to be at odds with the observation of universal critical behavior in QH systems.

In the present paper, these problems will be addressed and solved. In brief, the first one (indefinite metric) is overcome by trading  $\text{PSU}(1,1|2)$  for a submanifold of the Riemannian symmetric superspace  $\mathbf{X}$ , which is obtained by dividing out  $\mathbb{R}^+ \subset M_B$  and  $U(1) \subset M_F$ . The second one (central charge  $c \neq 0$ ) is rectified by postulating the existence of an auxiliary complex boson that can alternatively be regarded as forming part of the functional integration measure. The third one (noncriticality of all correlation functions that probe only the retarded or advanced sector) turns out to be resolved as a consequence of BRST invariance of the supersymmetric theory with properly defined target. Concerning the fourth point (marginal coupling  $f$ ), we shall argue that universality prevails to the extent that the conductivity governing the *classical* or incoherent transport near absorbing boundaries is universal.

Let us now summarize the plan of the paper. We start out by reviewing in Sections 2–4 three cornerstones of the theory of the QH plateau transition: the network model of Chalker and Coddington, Pruisken’s nonlinear sigma model (the supersymmetric version), and an antiferromagnetic superspin chain. The physical information drawn from them is condensed into a check list of conditions the fixed-point theory must satisfy, in Section 5. We then change gears and elucidate, in Section 6, the notions of Riemannian symmetric superspace and cs-manifold, which are needed for the definition of the target space and its invariant Berezin integral in Sections 7 and 8. A candidate for the fixed point is proposed in Section 9. Normalization of its partition function and triviality of the BRST invariant correlation functions is demonstrated in Section 10. Section 11 briefly reviews the arguments for conformal invariance of the model, and Section 12 checks more items of our list. The marginal coupling  $f$  is fixed in Section 13, by matching the short-distance singularity of the conductance between two interior contacts to the classical expectation. With the value of  $f$  thus determined, we argue in Section 14 that the algebraic decay of the typical point-contact conductance for the network model is governed by the irrational exponent  $X_t =$

$2/\pi$ , in agreement with the numerics. An assessment of where the theory now stands and where it will go, is given in the last section.

Finally, a word of warning is in order. This paper addresses an audience including disordered electron physicists, conformal field theorists, and high-energy physicists. Therefore, an effort was made to explain some trivial things. On the other hand, part of the material, particularly in the later sections, is too complex to be treatable below a certain minimum of mathematical sophistication, and some basic familiarity with the theory of symmetric spaces, supermanifolds, and harmonic analysis had to be assumed.

## 2 Network model

The purpose of the present paper is to propose a field-theory Lagrangian describing the critical behavior at the transition between two neighboring plateaus of the integer quantum Hall effect. Although our proposal is not constructive (in the sense of providing a complete sequence of steps leading from a microscopic model to the field theory), it will take a number of important clues and constraints from the representation of the quantum Hall universality class by noninteracting disordered electrons in a strong magnetic field. A particularly neat and efficient representative of this universality class is the network model of Chalker and Coddington, which we are now going to review.

In its original formulation [30], the model was conceived as a device for computing the transfer of electron wave amplitudes across a finite two-dimensional quantum Hall sample. A wave function of the model is defined to be a set of complex amplitudes, one for each edge or link of a square network. A characteristic feature, originating from the presence of a strong magnetic field, is the unidirectional motion specified by arrows, see Figure 1. The elementary building blocks of the model are  $2 \times 2$  scattering matrices  $S$  assigned to the vertices or nodes of the network. Being elements of the unitary group  $U(2)$ , these matrices can be written as

$$S = \begin{pmatrix} e^{i\varphi(o_1)} & 0 \\ 0 & e^{i\varphi(o_2)} \end{pmatrix} \begin{pmatrix} \cos \rho & \sin \rho \\ -\sin \rho & \cos \rho \end{pmatrix} \begin{pmatrix} e^{i\varphi(i_1)} & 0 \\ 0 & e^{i\varphi(i_2)} \end{pmatrix}.$$

Each phase factor  $e^{i\varphi}$  belongs to one link of the network. Disorder is introduced by taking the phase factors to be independent identically distributed random variables drawn from  $U(1)$ . Averages over the disorder will be denoted by  $\langle \dots \rangle$ . The distribution of the phase factors is taken to be uniform on  $U(1)$ , which results in the model having a local  $U(1)$  lattice gauge invariance. The parameter  $\rho$  is taken to be fixed (as opposed to random) and homogeneous over the network, and determines the probability for scattering to the left or right at each node to be  $p_L = |\cos \rho|^2$  or  $p_R = |\sin \rho|^2$ . The connection rules specified by the scattering

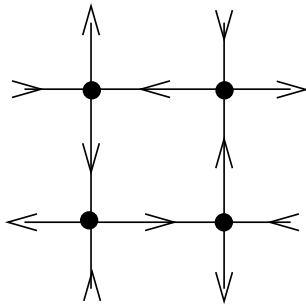


Figure 1: Unit cell of the network model.

matrices on the nodes define a transfer matrix for the total system. The network model is critical when the probabilities for scattering to the left and right are equal:  $p_L = p_R = 1/2$ . (We mention in passing that the approximations leading to the model are readily justified for slowly varying random potentials.)

Although the transfer matrix picture has its merits, it will be more useful for our purposes to think about the network model in another way, namely as a *dynamical* system with discrete time [31]. The evolution operator for one time step is a unitary operator denoted by  $U$ , and the dynamics is generated by iterating  $U$ . Thus, if the state of the electron at time  $t = 0$  is  $|\psi_0\rangle$ , the state after  $n$  elementary time steps is  $|\psi_{t=n}\rangle = U^n|\psi_0\rangle$ . The time evolution operator  $U$  is a product of two factors:  $U = U_0 U_1$ . The first of these ( $U_1$ ) encodes the deterministic part of the scattering at the nodes. The other factor ( $U_0$ ) multiplies the wave function on each link  $l$  by the corresponding random  $U(1)$  element  $e^{i\varphi(l)}$ .

When the network model is viewed as a quantum dynamical system, we can define for it an analog of the resolvent operator  $(E \pm i\epsilon - H)^{-1}$  of a Hamiltonian system with energy  $E$  and Hamiltonian  $H$ . Recalling the identity

$$(E + i\epsilon - H)^{-1} = -i \int_0^\infty e^{it(E+i\epsilon-H)} dt \quad (\epsilon > 0),$$

we do the following. We replace  $E + i\epsilon$  by a complex quasi-energy  $i^{-1} \ln z$  with  $|z| < 1$ , the time evolution operator  $e^{-itH}$  by  $U^n$ , and the integral over continuous time  $t$  by a sum over discrete time  $n$ . The above identity then transcribes to

$$(1 - zU)^{-1} = \sum_{n=0}^{\infty} z^n U^n.$$

Thus the correct analog of the operator  $(E + i\epsilon - H)^{-1}$  is  $(1 - zU)^{-1}$ , and the analog of  $(E - i\epsilon - H)^{-1}$  is  $(1 - \bar{z}\bar{U})^{-1}$ . Many observables of the network – as an example we mention the point-contact conductance [26] – can be expressed as products of matrix elements of these operators.

Although its primary realm of application is charge transport, the network model has also been profitably used for studying spectral correlations [32]. In



that case one takes the network to be *closed*, and views the eigenvalues of  $i \ln U$  as quasi-energy “levels” on the interval  $[0, 2\pi]$ . One quantity of interest is the so-called two-level correlation function  $R_2(\omega)$ , which has a well-known expression as the (discrete) Fourier transform of the quantum return probability  $\langle |\text{Tr } U^n|^2 \rangle$ :

$$R_2(\omega) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} \cos(n\omega) \langle |\text{Tr } U^n|^2 \rangle .$$

Another quantity that has been the focus of recent work [33] is the autocorrelation function of spectral determinants,

$$C(\omega) = \langle \text{Det}(1 - e^{i\omega/2} U) \text{Det}(1 - e^{i\omega/2} \bar{U}) \rangle .$$

More generally, we can consider the following correlation function:

$$\Omega(a_0, a_1; b_0, b_1) = \left\langle \frac{\text{Det}(1 - a_1 U) \text{Det}(1 - b_1 \bar{U})}{\text{Det}(1 - a_0 U) \text{Det}(1 - b_0 \bar{U})} \right\rangle . \quad (1)$$

We recover  $C(\omega)$  from it by setting  $a_0 = b_0 = 0$  and  $a_1 = b_1 = e^{i\omega/2}$ . We can also extract  $R_2(\omega)$ , by differentiating with respect to  $a_0$  and  $b_0$  at  $a_0 = a_1 = b_0 = b_1 = e^{i\omega/2}$  and then taking the real part. A (perturbed) conformal field theory formulation of the correlator  $\Omega$  at criticality will be proposed in Section 9. Here we wish to point out the following two features. Firstly, when  $a_0$  is set equal to  $a_1$ , or  $b_0$  equal to  $b_1$ , the correlator becomes trivial:

$$\Omega(a, a; b_0, b_1) = 1 = \Omega(a_0, a_1; b, b) .$$

Indeed, in the former case the first determinants in the numerator and denominator cancel, leaving  $\langle \text{Det}(1 - b_1 \bar{U}) / \text{Det}(1 - b_0 \bar{U}) \rangle$ . For  $|b_0| < 1$  this can be expanded in a convergent power series in  $\bar{U}$ . Since  $\langle e^{-in\varphi} \rangle = 0$  for  $n > 0$ , only the very first term in the series survives disorder averaging, giving the trivial result of unity. The same argument applies to the latter case ( $b_0 = b_1 = b$ ). For similar reasons, any correlation function or observable involving only retarded information ( $U^n$ ) or only advanced information ( $\bar{U}^n$ ) is trivial. This puts a strong constraint on any field theory that is to be a serious candidate for the QH plateau transition.

Secondly, recall that setting  $a_0 = b_0 = 0$  and  $a_1 = b_1 = e^{i\omega/2}$  yields the correlator  $C(\omega)$ . Its critical behavior has recently been argued [33] to fall in a much studied and well understood universality class that has an  $\text{SU}(2)$  invariance. This is the universality class of the six-vertex model with isotropic vertex weights, or the one-dimensional isotropic spin-1/2 Heisenberg antiferromagnet, or the sine-Gordon model at  $\beta^2 = 8\pi$ , or the  $xy$ -model at the Kosterlitz-Thouless temperature. The conformal field theory governing this class is known [25] to be the Wess-Zumino-Novikov-Witten (WZW) model of the group  $\text{SU}(2)$  at level

$k = 1$ . Thus another condition to impose on the theory we are looking for, is that it has to reduce to the  $SU(2)_1$  WZW model on sending  $\ln a_0$  and  $\ln b_0$  to minus infinity.

While the derivation given in [33] took the thorny route of the  $O(3)$  nonlinear sigma model, the critical nature of the correlator  $C(\omega)$  can be understood quite directly and convincingly from the network model. Let us take a break from the general development and provide a few details here. We start off by representing the product of spectral determinants as a Gaussian integral over (retarded and advanced) anticommuting fields  $\psi_\uparrow(l)$  and  $\psi_\downarrow(l)$  placed on the links  $l$  of the network. The average over the random phase factors  $e^{i\varphi(l)}$  is then carried out by using at every link the identity

$$\int_0^{2\pi} d\varphi \exp(e^{i\varphi} \bar{\psi}_\uparrow \psi_\uparrow + e^{-i\varphi} \bar{\psi}_\downarrow \psi_\downarrow) = 2 \int_{\mathbb{C}} \frac{dz d\bar{z}}{(1 + \bar{z}z)^3} \exp(z \bar{\psi}_\uparrow \psi_\downarrow - \bar{z} \bar{\psi}_\downarrow \psi_\uparrow) ,$$

which is a special case of the “color-flavor transformation” [34] and is elementary to verify by Taylor expansion of the integrand on both sides. A beautiful feature of the transformation is that it preserves the Gaussian dependence on  $\psi$ . We can therefore integrate out  $\psi$  again, and arrive at another determinant, now with matrix entries that depend on the complex field  $z(l)$ . By the structure of the network model, the determinant factorizes as a *product* of terms, or weights, one for each vertex of the network. At the critical point  $p_L = p_R = 1/2$ , and for a vertex with incoming links  $i_1$  and  $i_2$ , and outgoing links  $o_1$  and  $o_2$ , the weight is [35]

$$\frac{1 + \frac{1}{2} e^{i\omega} (\bar{z}(o_1) + \bar{z}(o_2)) (z(i_1) + z(i_2)) + e^{2i\omega} \bar{z}(o_1) \bar{z}(o_2) z(i_1) z(i_2)}{\sqrt{1 + \bar{z}(o_1) z(o_1)} \sqrt{1 + \bar{z}(o_2) z(o_2)} \sqrt{1 + \bar{z}(i_1) z(i_1)} \sqrt{1 + \bar{z}(i_2) z(i_2)}} .$$

Hence the transformed theory has the structure of a *vertex model*. To bring it into a familiar form, one has to recognize the intrinsic meaning of the complex number  $z(l)$  (for a fixed link  $l$ ) as a parameter for coherent spin-1/2 states:  $|\uparrow\rangle + z|\downarrow\rangle$ . (This meaning is particularly evident from the general proof [36] of the color-flavor transformation.) By interpreting the vertex weights as matrix elements between coherent states, and using the closure relation

$$\frac{1}{\pi} \int_{\mathbb{C}} \frac{dz d\bar{z}}{(1 + \bar{z}z)^3} \left( |\uparrow\rangle + z|\downarrow\rangle \right) \left( \langle\uparrow| + \bar{z}\langle\downarrow| \right) = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| ,$$

one can rewrite the correlator  $C(0)$  as the partition sum of a vertex model for classical variables taking two values,  $\uparrow$  and  $\downarrow$ . The vertex weights can be extracted from the above expression by observing that  $z = 0$  means spin up, and  $z = \infty$  spin down. For  $\omega = 0$  the vertex weights are isotropic, or  $SU(2)$ -symmetric, giving a model that belongs to a family known as the six-vertex model in the area of integrable systems [37]. Note that the calculation we have sketched

is free of approximations, so the autocorrelation function of spectral determinants  $C(0)$  at the critical point is *exactly equal* to the partition sum of the six-vertex model at its  $SU(2)$ -symmetric point. The parameter  $\omega$  acts as a perturbation breaking  $SU(2)$  symmetry as well as criticality. (In the anisotropic limit of the six-vertex model as an  $xxx$  quantum spin chain with Hamiltonian  $H$ , the system is perturbed by coupling to a homogeneous imaginary field,  $H \rightarrow H + i\omega \sum_n S_n^z$ .)

While the equivalence between the network model correlator  $C(\omega)$  and a (perturbed) six-vertex model is interesting in its own right and deserves further study, we will not pursue it here, as it does not form the subject of the present paper. We have pointed out the equivalence because it adds much support to the claim [33] that one basic building block of the field theory we are seeking is the conformal field theory limit of the six-vertex model, namely the  $SU(2)_1$  WZW model.

### 3 Pruisken's nonlinear sigma model

Historically, the network model of Chalker and Coddington was preceded by Pruisken's nonlinear sigma model, which dominated the early efforts to understand electron delocalization in the integer quantum Hall effect. This important cornerstone of the theory will be reviewed next.

The task of computing transport coefficients for a disordered electron system amounts to calculating disorder averages of products of retarded and advanced electron Green functions. A good way of going about it is to map the problem on an effective field theory. The initial development [38, 39, 40] of the subject relied on the replica trick for computing disorder averages. For the case of systems in a *weak* magnetic field, this led to a nonlinear sigma model with target space  $G/H$ , where  $G/H = U(n, n)/U(n) \times U(n)$  or  $U(2n)/U(n) \times U(n)$  with  $n = 0$  for bosonic resp. fermionic replicas. A mathematically satisfactory variant based on the supersymmetric formalism emerged from the pioneering work of Efetov [9]. In that formulation,  $G = U(1, 1|2)$ , the group of pseudo-unitary  $4 \times 4$  supermatrices  $g$  preserving an indefinite Hermitian form:

$$g^\dagger \eta g = \eta, \quad \eta = \text{diag}(-1, +1, +1, +1),$$

and  $H = U(1|1) \times U(1|1)$ . (For many purposes, it is preferable to work with the complexified group  $G_{\mathbb{C}} = GL(2|2)$  but we will stick with  $G$  for now.) The Lagrangian of the theory is conventionally presented in terms of a field denoted by  $Q$ :

$$Q = g \Sigma_3 g^{-1}, \quad \Sigma_3 = \text{diag}(+1, +1, -1, -1).$$

The diagonal matrix  $\Sigma_3$  discriminates between the retarded (+) and advanced

( $-$ ) sector of the theory. A natural parametrization of the nonlinear field  $Q$  is

$$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1},$$

where

$$Z = \begin{pmatrix} Z_{\text{BB}} & Z_{\text{BF}} \\ Z_{\text{FB}} & Z_{\text{FF}} \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \tilde{Z}_{\text{BB}} & \tilde{Z}_{\text{BF}} \\ \tilde{Z}_{\text{FB}} & \tilde{Z}_{\text{FF}} \end{pmatrix} \quad (2)$$

are complex  $2 \times 2$  supermatrices, with  $\tilde{Z}_{\text{FF}} = -\bar{Z}_{\text{FF}}$  and  $\tilde{Z}_{\text{BB}} = +\bar{Z}_{\text{BB}}$ . (We do not specify how complex conjugation relates  $Z_{\text{BF}}, Z_{\text{FB}}$  to  $\tilde{Z}_{\text{FB}}, \tilde{Z}_{\text{BF}}$ , as there is no need for that.) The variable  $Z_{\text{FF}}$  takes values in  $\mathbb{C}$  with a point added at infinity, which is the same as a two-sphere  $S^2 \simeq \text{U}(2)/\text{U}(1) \times \text{U}(1)$ . The range of  $Z_{\text{BB}}$  is restricted by  $|Z_{\text{BB}}|^2 < 1$ . This is Poincaré's model of the two-hyperboloid  $H^2 \simeq \text{U}(1, 1)/\text{U}(1) \times \text{U}(1)$ . The Lagrangian of the field theory is

$$\begin{aligned} L_0 &= -\frac{\sigma_{xx}}{8} \text{STr} \partial_\mu Q \partial_\mu Q \\ &= \sigma_{xx} \text{STr} (1 - \tilde{Z}Z)^{-1} \partial_\mu \tilde{Z} (1 - Z\tilde{Z})^{-1} \partial_\mu Z, \end{aligned}$$

where  $\text{STr}$  means the supertrace. The coupling constant  $\sigma_{xx}$  has an interpretation as the dissipative conductivity of the electron gas (conductances being measured in natural units  $e^2/h$ ). In two space dimensions, the coupling  $\sigma_{xx}$  is dimensionless, and from renormalization-group assisted perturbation theory one expects the existence of a mass gap, implying that all electron states are localized in that case.

Pruisken's insightful contribution [7] was to add to the Lagrangian a topological density,  $L_{\text{top}}$ . Such a term exists, and is nontrivial, both for the fermionic replica theory and for the supersymmetric theory. The latter version was first formulated by Weidenmüller [8], and reads

$$\begin{aligned} L_{\text{top}} &= \frac{\sigma_{xy}}{8} \epsilon_{\mu\nu} \text{STr} Q \partial_\mu Q \partial_\nu Q \\ &= \sigma_{xy} \epsilon_{\mu\nu} \text{STr} (1 - \tilde{Z}Z)^{-1} \partial_\mu \tilde{Z} (1 - Z\tilde{Z})^{-1} \partial_\nu Z. \end{aligned}$$

The coupling constant  $\sigma_{xy}$  is identified with the Hall conductivity of the two-dimensional electron gas. Pruisken's key idea was that such a term is needed to break parity (or reflection of the plane of the electron gas), which is not a symmetry in the presence of a strong magnetic field, and will cause critical behavior, or delocalization of the electrons, for  $\sigma_{xy} \in \mathbb{Z} + 1/2$ . The term is called topological because it arises from pulling back a closed two-form (the Kähler form) on  $G/H$ . Its integral over two-cycles therefore takes quantized values,  $2\pi n i \sigma_{xy}$  with  $n \in \mathbb{Z}$ , and the Hall conductivity acquires the meaning of a topological angle  $\theta = 2\pi \sigma_{xy}$ .

To generate specific observables of the two-dimensional electron gas, one includes sources in the Lagrangian and takes derivatives as usual. For example, the correlator of spectral determinants  $\Omega(a_0, a_1; b_0, b_1)$  is obtained by adding a term

$$L_\omega = \Lambda^2 \text{STr}(\omega Q - \omega \Sigma_3) ,$$

where  $\omega = \text{diag}(-\ln a_0, -\ln a_1, \ln b_0, \ln b_1)$  and  $\Lambda$  is an ultraviolet cutoff. The parameters  $a_0$  and  $b_0$  are subject to

$$\text{Re} \ln a_0 < 0 , \quad \text{Re} \ln b_0 < 0 .$$

They act as regulators for the zero modes of the hyperbolic degrees of freedom ( $Z_{\text{BB}}$ ) of the matrix field  $Q$ .

For many years the disordered electron community has debated whether Pruisken's model is the "correct" or "good" theory of the integer quantum Hall plateau transition. The issue is not whether the model for  $\sigma_{xy} = 1/2$  is massless (it certainly is), but whether on the critical line  $\sigma_{xy} = 1/2$  there exists a renormalization group fixed point where the beta function for  $\sigma_{xx}$  vanishes. From the practical point of view, one also wants to know whether one can analytically solve [41] such a fixed-point theory, if it exists. These questions have never been answered conclusively. At weak coupling ( $\sigma_{xx} \gg 1$ ), the beta function can be computed by perturbation theory. The topological term  $L_{\text{top}}$  is perturbatively invisible, and hence the situation is the same as for  $\sigma_{xy} = 0$ : the one-loop beta function vanishes, as the geometry of the (symmetric) space  $G/H$  is Ricci flat, but in two-loop order the quantum fluctuations kick in and drive the theory towards strong coupling. On the basis of dilute instanton gas calculations, Pruisken and collaborators [11] argued that nonperturbative effects due to the topological term terminate the RG flow at some finite coupling  $\sigma_{xx}^*$ . However, close scrutiny shows that the argument is not really convincing, as the instanton gas can only be controlled, if at all, at weak coupling and a far extrapolation to strong coupling is required. Thus the status of Pruisken's model as a candidate for the fixed-point theory has remained unclear.

However, a certain bias *against* Pruisken's model came from the following observation. When the parameters  $\ln a_0$  and  $\ln b_0$  are moved to minus infinity, the fields  $Z_{\text{BB}}$ ,  $Z_{\text{BF}}$ , and  $Z_{\text{FB}}$  become massive and drop out of the theory, leaving behind a massless sector  $S^2 \simeq \text{U}(2)/\text{U}(1) \times \text{U}(1)$  governed by the Lagrangian

$$L = \frac{1}{g^2} \frac{\partial_\mu \bar{z} \partial_\mu z}{(1 + \bar{z}z)^2} + \frac{\theta \epsilon_{\mu\nu}}{2\pi} \frac{\partial_\mu \bar{z} \partial_\nu z}{(1 + \bar{z}z)^2} + \Lambda^2 \ln(a_1 b_1) \frac{1 - \bar{z}z}{1 + \bar{z}z} ,$$

where  $z \equiv Z_{\text{FF}}$ ,  $1/g^2 = \sigma_{xx}$ , and  $\theta = 2\pi\sigma_{xy}$ . This is the Lagrangian of the so-called O(3) nonlinear sigma model with a topological term. It has a global SU(2) [or O(3)] symmetry, perturbed by the symmetry-breaking field  $\ln(a_1 b_1)$ . Notice that on the critical line  $\sigma_{xy} = 1/2$ , the topological angle  $\theta$  equals  $\pi$ .

There exists a general consensus about the fate under renormalization of the  $O(3)$  model at  $\theta = \pi$ . Perturbation theory shows that the coupling constant of this field theory, like Pruisken's model, increases under renormalization. (It already does in one-loop order.) In the strong-coupling limit  $g^2 \rightarrow \infty$ , the low-energy Hamiltonian of the lattice-regularized theory becomes [42] the Hamiltonian of the 1d Heisenberg model of an  $SU(2)$ -invariant quantum antiferromagnet with spin  $S = 1/2$ . Recall that we arrived at the same theory already in Section 2, by starting from the network model and pointing out that the critical correlator  $\Omega(0, a_1; 0, b_1)$  maps on the partition function of a six-vertex model, which in turn has the Heisenberg antiferromagnet for its (spatially) anisotropic limit. As was mentioned earlier, the low-energy physics of the latter is governed by the  $SU(2)_1$  WZW model, perturbed by a current-current interaction. The perturbation is marginally irrelevant, and the theory in the infrared flows to a conformal invariant fixed point, where the global  $SU(2)$  invariance of the nonlinear sigma model is promoted to an  $SU(2)_L \times SU(2)_R$  Kac-Moody symmetry.

By analogy, one expects a similar scenario to take place for Pruisken's model at  $\sigma_{xy} = 1/2$  and  $\omega = 0$ : global invariance under the complexification  $G \equiv SL(2|2)$  of  $SU(1, 1|2)$  should be promoted to a chiral symmetry  $G_L \times G_R$  in the fixed-point theory. The argument for symmetry doubling will be reviewed in the next section, after the introduction of its essential ingredient, namely a “superspin” analog (for Pruisken's model) of the spin-1/2 Heisenberg chain.

## 4 Superspin chain

We start out with a concise description of the superspin chain, and will indicate the relation to the Chalker-Coddington network and Pruisken's nonlinear sigma model afterwards. The chain is a one-dimensional “antiferromagnet” with degrees of freedom that take values in an alternating sequence of  $\mathfrak{gl}(2, 2)$  modules  $V$  and  $V^*$ . Our first task is to describe these modules. They were first identified in unpublished work by N. Read [43].

To begin, let  $E_{ij}$  denote the matrix whose entries are zero everywhere except at the intersection of the  $i$ -th row with the  $j$ -th column where the entry is unity. By  $\mathfrak{gl}(2, 2)$  we mean the Lie superalgebra spanned by  $\{E_{ij}\}_{i,j=0,\dots,3}$  over  $\mathbb{C}$ , with the bracket or supercommutator given by

$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - (-1)^{(i+j)(k+l)} E_{kl}E_{ij} .$$

Next put

$$\begin{aligned} \bar{c}_0 &= b_+^\dagger, & \bar{c}_1 &= f_+^\dagger, & \bar{c}_2 &= -b_- , & \bar{c}_3 &= f_- , \\ c_0 &= b_+ , & c_1 &= f_+ , & c_2 &= b_-^\dagger , & c_3 &= f_-^\dagger , \end{aligned}$$

where  $b_\pm^\dagger, f_\pm^\dagger$  and  $b_\pm, f_\pm$  are creation and annihilation operators for “charged”  $(\pm)$  bosons and fermions. They obey the canonical commutation and anticom-

mutation relations, and act in a Fock space  $\mathcal{F}$  with vacuum  $|0\rangle$ . The mapping

$$E_{ij} \mapsto S_{ij} \equiv \bar{c}_i c_j$$

determines a representation of  $\mathfrak{gl}(2, 2)$  on  $\mathcal{F}$ . According to general theory [44], this representation nicely decomposes as a *direct sum of irreducibles*. The irreducible representation spaces are labelled by an integer, which is the eigenvalue of the operator  $C = \sum_i \bar{c}_i c_i$  generating the *center* of  $\mathfrak{gl}(2, 2)$ . It is evident that  $C$  counts the difference between the number of positively and negatively charged particles (or of “retarded” and “advanced” particles, using the terminology of the disordered electron system). Thus  $C$  is the total charge. We want the irreducible space,  $V$ , on which  $C$  vanishes. Note that  $V$  contains the vacuum  $|0\rangle$ . The latter actually is a lowest-weight state for the  $\mathfrak{gl}(2, 2)$  module  $V$ .

The definition of the conjugate module  $V^*$  completely parallels that of  $V$ , except that the fundamental identifications change to

$$\begin{aligned} \bar{c}_0 &= -b_+ , & \bar{c}_1 &= f_+ , & \bar{c}_2 &= b_-^\dagger , & \bar{c}_3 &= f_-^\dagger , \\ c_0 &= b_+^\dagger , & c_1 &= f_+^\dagger , & c_2 &= b_- , & c_3 &= f_- . \end{aligned}$$

(Equivalently, we could keep the operator identifications, and alternate the definition of the vacuum state [35].) Both modules  $V$  and  $V^*$  are infinite-dimensional, since we can keep creating boson pairs  $b_+^\dagger b_-^\dagger$  with no limit. They are naturally completed as Hilbert spaces with the usual Hermitian scalar product that makes  $b^\dagger$  the adjoint of  $b$ , and  $f^\dagger$  the adjoint of  $f$ . Adopting the spin terminology used in Section 2, we refer to the elements of  $V$  and  $V^*$  as the state vectors of a “superspin”.

The superspin chain is now defined as an alternating sequence,

$$\dots \otimes V \otimes V^* \otimes V \otimes V^* \otimes V \otimes V^* \otimes \dots ,$$

of an even number  $N$  of superspins, with Hamiltonian

$$\mathcal{H} = \sum_n \sum_{i,j=0}^3 S_{ij}(n) (-1)^{j+1} S_{ji}(n+1)$$

at the critical point. The bilinear expression  $\sum_{ij} S_{ij} (-1)^j S_{ji}$  represents the quadratic Casimir invariant of  $\mathfrak{gl}(2, 2)$ . To move the system off criticality, one staggers the coupling between the sites  $n$ . The alternation of the space of states between  $V$  and  $V^*$  renders the chain “antiferromagnetic” in character. Indeed, the two-superspin system  $V \otimes V^*$  with Hamiltonian  $\mathcal{H}$  has been shown [45] to have a  $\mathfrak{gl}(2, 2)$  invariant ground state with zero energy (separated by a gap from a continuum of excited states) as befits the supersymmetric generalization of a quantum antiferromagnet.

Before carrying on, we insert the following technical remark. The Lie superalgebra  $\mathfrak{gl}(2, 2)$  suffers from the disease of being non-semisimple. It has a one-dimensional center generated by the unit matrix  $I \equiv \sum_i E_{ii}$ , which obviously commutes with all matrices. At the same time, since  $\text{STr} [X, Y] = 0$  for any pair  $X, Y \in \mathfrak{gl}(2, 2)$ , there exists a one-dimensional subspace of elements that never appear on the right-hand side of any supercommutation relation. This is the subspace of multiples of the generator with nonvanishing supertrace,  $F \equiv \sum_i (-1)^i E_{ii}$ . Removal of the latter defines the subalgebra  $\mathfrak{sl}(2, 2) \subset \mathfrak{gl}(2, 2)$ . One can also take the quotient by the center and pass to  $\mathfrak{psl}(2, 2) = \mathfrak{sl}(2, 2)/\mathbb{C} \cdot I$ , which is still a Lie superalgebra. The quadratic Casimir decomposes as

$$\mathcal{C}_{\mathfrak{gl}(2,2)} = \mathcal{C}_{\mathfrak{psl}(2,2)} + \text{const} \times IF .$$

Since  $I$  is represented by charge  $C = 0$  on  $V$  and  $V^*$ , it is possible to express the Hamiltonian  $\mathcal{H}$  of the superspin chain completely in terms of the generators of  $\mathfrak{psl}(2, 2)$ . However, for the purposes of the present section we find it convenient not to project on  $\mathfrak{psl}(2, 2)$ , but rather to tolerate the presence of a nontrivial center and work with  $\mathfrak{sl}(2, 2)$ . (The center will then give rise to gauge degrees of freedom in the field theory.)

Given the Lie superalgebra representation  $E_{ij} \mapsto \bar{c}_i c_j$ , we can exponentiate to obtain a  $\text{SL}(2|2)$  group action on  $V$  (or  $V^*$ , it works the same way in both cases) by

$$g \mapsto \exp \left( \sum_{ij} \bar{c}_i (\ln g)_{ij} c_j \right) .$$

For this action to make sense, we have to allow multiplication of the state vectors of  $V$  and  $V^*$ , now viewed as  $\mathfrak{sl}(2, 2)$  modules, by the anticommuting parameters that are needed to form the supermatrix  $g \in \text{SL}(2|2)$ . Because the eigenvalues of  $\bar{c}_i c_i$  are integers, the exponential is well-defined in spite of the multi-valuedness of the logarithm. The complex group  $\text{SL}(2|2)$ , of course, does not act by isometries on the Hilbert spaces  $V$  and  $V^*$ , but there exists a bosonic subgroup  $\text{SU}(1, 1) \times \text{SU}(2)$  which does. In other words,  $V$  and  $V^*$  decompose into *unitary* representations of  $\text{SU}(1, 1) \times \text{SU}(2)$ . The pseudo-unitary subgroup  $\text{SU}(1, 1|2) \subset \text{SL}(2|2)$  (the symmetry group of Pruisken's nonlinear sigma model) is less useful here, as some of its odd generators do not act unitarily (or create "states with negative norm"). The  $\text{SL}(2|2)$  action on  $V$  and  $V^*$  extends to an action of  $\text{SL}(2|2)^N$  on the superspin chain of length  $N$ . Since  $\sum_{ij} S_{ij} (-1)^j S_{ji}$  is the quadratic Casimir invariant of  $\mathfrak{gl}(2, 2)$  represented on  $V$  or  $V^*$ , the diagonal of  $\text{SL}(2|2)^N$  (the global action) commutes with the superspin Hamiltonian  $\mathcal{H}$ . This completes our definition of the superspin chain and its symmetries.

There exist several ways of arriving at the superspin chain and its Hamiltonian. The first published derivation [45] took Pruisken's nonlinear sigma model for its starting point. Following the treatment by Shankar and Read [42] of the



O(3) nonlinear sigma model at  $\theta = \pi$ , the two-dimensional supersymmetric field theory was discretized on a lattice of sites in the spatial direction and put in one-dimensional quantum Hamiltonian form. For strong coupling, the “large” part of the Hamiltonian is site-diagonal, with the single-site Hamiltonian being  $(p - A)^2$ , where  $p^2$  is the Laplacian on the target space  $U(1, 1|2)/U(1|1) \times U(1|1)$  of the nonlinear sigma model, and  $A$  is the gauge field of a fictitious magnetic monopole. The single-site Hamiltonian has zero-energy states, which are naturally described as holomorphic sections of an associated line bundle. For the case of an even-numbered site, the vector space of these zero modes is precisely the module  $V$ , and for odd sites it is  $V^*$ . The degeneracy between the zero-energy states is lifted by the coupling between sites, and this interaction projects on the superspin Hamiltonian  $\mathcal{H}$  written down above. Thus we conclude that  $\mathcal{H}$  governs the low-energy physics of Pruisken’s model at strong coupling.

The superspin Hamiltonian can be obtained more easily by starting from the spatially anisotropic limit of the Chalker-Coddington model, which is a sequence of counterpropagating “edges” coupled by random complex tunneling amplitudes. Interpreting the functional integral representation of the disorder averaged theory as a coherent-state path integral, and passing to a Hamiltonian description with the help of the transfer matrix, one arrives quite directly (and without having to make approximations) at the spin chain. This was first done using replicas by D.-H. Lee [46]. A supersymmetric version was given in [35, 47].

The important conclusion of all this is that Pruisken’s nonlinear sigma model at strong coupling is equivalent to the Chalker-Coddington model, and both can be represented as a superspin chain. Thus there is a convergence of models and formulations, and the focus now is on the antiferromagnetic superspin chain. There exists some recent numerical work using the density-matrix renormalization group, along with analytical ideas motivated by the Lieb-Schultz-Mattis theorem, which confirm the expectation that the superspin chain is quantum critical [48]. There exist also attempts [49] to deform the superspin chain to an integrable model that is solvable in the Yang-Baxter sense. Unfortunately, the technical difficulties encountered there are enormous, and an analytical solution does not seem to be within close reach.

Although the antiferromagnetic superspin chain at low energies is not easy to control, we can still make some useful *qualitative* predictions from it, as follows. Recall that the action of  $G = SL(2|2)$  on  $V$  and  $V^*$  gives an action of  $G^N$  on the chain of length  $N$ . If  $g = g_1 \times g_2 \times \dots \times g_N \in G^N$  (Cartesian product), we write  $\hat{g} = \hat{g}_1 \times \hat{g}_2 \times \dots \times \hat{g}_N$  for the element acting on the chain. The ground state of the chain with Hamiltonian  $\mathcal{H}$  is some complicated object,  $|\psi_0\rangle$ . We do not know how to describe it analytically, but it certainly exists, and since the superspin chain is antiferromagnetic, we can be sure that  $|\psi_0\rangle$  is *invariant* with respect to the global  $G$  action. (The ground state of a 1d isotropic antiferromagnet with an even number of spins is always a spin singlet.) In formulas:  $\hat{g}|\psi_0\rangle = |\psi_0\rangle$  for any

global  $\hat{g} = \hat{g}_1 \times \hat{g}_1 \times \dots \times \hat{g}_1$ . The  $G$  symmetry of the Hamiltonian is expressed by the equation  $\hat{g}\mathcal{H}\hat{g}^{-1} = \mathcal{H}$ , again for global  $\hat{g}$ .

Recall next that  $V$  and  $V^*$  are *irreducible*  $\mathfrak{sl}(2, 2)$  modules. Irreducibility of  $V$  and  $V^*$  means that *all* state vectors of the superspin chain are reached by acting with  $G^N = \mathrm{SL}(2|2)^N$  on some reference state. Now, what we are after is the *low-energy physics* of the chain. For a chain with a large correlation length, the low-energy states are generated by acting with “slowly varying”  $\hat{g}$  on the ground state  $|\psi_0\rangle$ . In other words, we expect the existence of a continuum limit where the low-energy states form an irreducible module for a *loop group*,  $LG$ . (For this we impose periodic boundary conditions on the chain). The elements of the loop group  $LG$  are smooth maps  $S^1 \rightarrow G$ , denoted by  $\hat{g}(x)$ . The action  $g \mapsto \hat{g}$  of  $G^N$  on the discrete chain carries over to a representation  $g(x)h(x) \mapsto \hat{g}(x)\hat{h}(x)$  of the loop group, in the continuum limit.

Given the action of the loop group, we can form  $LG$ -coherent states  $\hat{g}|\psi_0\rangle$ . Let us now postulate the existence of a functional integral measure  $D\hat{g}$ , such that

$$\int D\hat{g} \hat{g}|\psi_0\rangle\langle\psi_0|\hat{g}^{-1} = \mathrm{id}$$

is a resolution of unity for the low-energy sector of the superspin chain. Using it we can pass to a  $LG$ -coherent state path integral [50], with the functional integrand being a product of matrix elements of the form

$$\langle\psi_0|\hat{g}_{\tau+\delta\tau}^{-1} e^{-\epsilon\mathcal{H}}\hat{g}_\tau|\psi_0\rangle. \quad (3)$$

If we knew  $|\psi_0\rangle$  and knew how to compute such matrix elements, we would have a constructive way of deriving the low-energy effective field theory of the QH plateau transition, which will be a functional integral over continuous fields  $g(x, \tau)$ . Since we know neither, the present argument remains formal.

Nevertheless, the  $LG$ -coherent state path integral exists in principle and leads us to the following assertion: the functional integral for  $g(x, \tau)$  has a *chiral* symmetry  $G_L \times G_R$ . Indeed, since  $g(x) \mapsto \hat{g}(x)$  is a representation, the matrix element (3) does not change under global transformations  $g_\tau(x) \mapsto g_L g_\tau(x) g_R$  and  $g_{\tau+\delta\tau}(x) \mapsto g_L g_{\tau+\delta\tau}(x) g_R$ . The left invariance is due to invariance of the Hamiltonian, the right invariance is a result of invariance of the ground state.

The argument given so far can be made regardless of whether the chain is critical or not. (The Hamiltonian is always invariant, and so is the ground state.) Now comes an important distinction. Away from criticality, where the correlation length is finite, the system has a *mass gap*. Therefore, the low-energy phase space must be reduced in some manner. Technically speaking, the loop group does not act freely (*i.e.*, not all states created by its generators are linearly independent), and hence some its degrees of freedom need to be gauged out in the functional integral. Gauging happens on the right (where the loop group acts on states) and thus interferes with the symmetry action on the right, leaving only the

left action as a good symmetry of the theory. (This is the situation in Pruisken's nonlinear sigma model.) On the other, in the massless theory we do expect the loop group to act freely, modulo the global invariance  $\hat{g}|\psi_0\rangle = |\psi_0\rangle$  leading to the phase space being  $LG/G$ . In that case the naive  $G_L \times G_R$  invariance of the coherent-state path integral should exist as a true symmetry of the properly defined field theory. This then is our main conclusion: in the fixed-point theory, the global  $G$  symmetry of Pruisken's model is promoted to a *larger* symmetry, namely an invariance under two copies of the symmetry group,  $G_L \times G_R$ .

To avoid confusion, let me emphasize that we are not arguing in favor of a stronger statement due to Affleck [25]. That argument says that for *unitary* theories with continuous  $G$  symmetry the fixed point acquires a Kac-Moody symmetry, which is *local* and hence infinite-dimensional. All we get from the above argument is a chiral doubling of the *global* symmetry group. (In order for the symmetry to become Kac-Moody, the field  $g(x, \tau)$  would have to separate into left-moving and right-moving waves. We will see later that this does not happen in the present case.)

There exists a second piece of valuable information we can infer from the  $LG$ -coherent state path integral of the superspin chain. Recall that  $g \in \text{SL}(2|2)$  acts on  $V$  and  $V^*$  by  $\exp \sum_{ij} \bar{c}_i (\ln g)_{ij} c_j$ . Evidently the multiples of unity,  $g = e^s \cdot 1_4$ , couple to the total charge  $C = \sum_i \bar{c}_i c_i$ . By construction,  $C = 0$  on every state vector of  $V$  and  $V^*$ , and therefore  $\sum_i \bar{c}_i(n) c_i(n)$  *vanishes on all states* of the superspin chain. As a result, nothing depends on the coherent state parameter  $s_\tau(n)$  conjugate to the local charge  $\sum_i \bar{c}_i(n) c_i(n)$ . In the continuum limit, this independence becomes an invariance of the  $LG$ -coherent state path integral under local gauge transformations  $g(x, \tau) \mapsto e^{s(x, \tau)} g(x, \tau)$  where  $e^{s(x, \tau)} \in \text{GL}(1)$  is any invertible multiple of the unit matrix. This symmetry can be seen to be a direct consequence of the local  $\text{U}(1)$  gauge invariance of the Chalker-Coddington model [ $\text{GL}(1)$  here is precisely the complexification of  $\text{U}(1)$ ].

## 5 Constraints on the fixed-point theory

We have reviewed the argument why we are going to abandon Pruisken's model and look for another field theory with manifest conformal invariance. Ideally, we would like to deduce the theory *constructively*, by starting from a model of disordered electrons such as the network model of Chalker and Coddington, and making controlled approximations. Unfortunately, this looks like a rather difficult if not impossible project, as the fixed point always appears to be at strong coupling, no matter what choice of starting point is made (the network model, the nonlinear sigma model, or a model of disordered Dirac fermions [22]). In such a situation, we are forced to resort to indirect reasoning. The viable procedure is to make an educated guess, and verify its correctness by comparing the consequences to known results. In making such a guess, we are guided by the

fact that the theory we are looking for meets the following list of requirements.

1. The field theory is defined by a stable functional integral with a target space of Euclidian signature (as opposed to Lorentzian or other signature). This condition eliminates, in particular, target spaces such as  $\mathrm{PSL}_{\mathbb{R}}(2|2)$  or  $\mathrm{PSU}(1,1|2)$ , which have appeared in recent work motivated by string propagation in  $\mathrm{AdS}_3$  backgrounds [28, 29].

2. Conformal invariance of the fixed-point theory is manifest. The energy-momentum tensor splits into a holomorphic and an antiholomorphic piece, the Fourier components of which obey the commutation relations of a Virasoro algebra. (Note that it is *not* obligatory for the theory have an extended symmetry algebra of the Kac-Moody type.)

3. The partition function of the theory is normalized to unity:  $Z = 1$ , independent of the size or other parameters of the system. For a conformal field theory this implies that the Virasoro central charge  $c$  vanishes. An immediate consequence is that the theory cannot be unitary, as this would require  $c > 0$ .

4. The field-theory representation of the correlation function (1) reduces to unity on setting either  $a_0 = a_1$  or  $b_0 = b_1$ . More generally, correlation functions and other observables of the theory become trivial when only one causal sector (retarded or advanced) is probed.

5. The Hamiltonian of the antiferromagnetic superspin chain is invariant under a global action of  $G = \mathrm{SL}(2|2)$ . In the fixed-point Lagrangian, this invariance is promoted to a (global) chiral symmetry  $G_L \times G_R$ . Invariance under the center of  $G$ , which is the subgroup generated by the unit matrix, is present as a *local* gauge symmetry.

6. Massive perturbation of the Boson-Boson sector (or equivalently, sending the parameters  $\ln a_0$  and  $\ln b_0$  to minus infinity) preserves criticality of the theory, by leaving some of the bosonic fields massless. The reduced theory for the massless modes is the  $\mathrm{SU}(2)_1$  WZW model (not necessarily at its conformal invariant fixed point).

7. The theory contains an operator corresponding to the density of states of the disordered electron system, the scaling dimension of which equals zero. This requirement follows from the known fact [51] that the density of states at the critical point under consideration is noncritical.

8. The fixed-point theory reproduces the critical exponents for various correlation functions that are known from numerical and real experiments.

In the sequel, we will describe a field theory which demonstrably satisfies all of the requirements 1–7. Since the theory turns out to be of a novel kind, with solutions not being readily available, requirement 8 can only be partially verified at the present time and will need further work in the future.

## 6 Symmetric superspaces and cs-manifolds

The correct definition of the target space of the field theory relies on the two notions of Riemannian symmetric superspace and cs-manifold. They are not entirely standard and are not commonly understood. It is therefore proper to put the general development on hold, and motivate and explain these notions of supermanifold theory in some detail, which is what we will do below.

The base of the supersymmetric target space  $G/H$  of Pruisken's model is  $H^2 \times S^2$ . As we recalled, the nonlinear sigma model with target space  $S^2$  (the  $O(3)$  model) and topological angle  $\theta = \pi$  flows under renormalization to a theory of fields taking values in  $SU(2) \simeq S^3$ . Thus quantum fluctuations in this case promote the two-sphere to the three-sphere. From that we take it that we should try promoting the noncompact sector  $H^2$  of Pruisken's model to a three-hyperboloid  $H^3$ .

The first question to address then is whether there exist conformal field theories of  $H^3$ -valued fields. The answer is yes.  $H^3$  is diffeomorphic to the noncompact symmetric space  $SL(2, \mathbb{C})/SU(2)$ , which in turn can be identified with the positive Hermitian unimodular  $2 \times 2$  matrices  $M$ , by setting  $M = gg^\dagger$  and letting  $g$  run through  $SL(2, \mathbb{C})$ . For the latter we can write down a WZW Lagrangian of the usual form

$$L = -(2\pi f^2)^{-1} \text{Tr} \partial M \bar{\partial} M^{-1} + (ik/12\pi) d^{-1} \text{Tr} (M^{-1} dM)^{\wedge 3}.$$

The partial derivatives  $\partial$  and  $\bar{\partial}$  are the usual ones, *i.e.*, if  $z$  and  $\bar{z}$  are complex coordinates for two-dimensional space (or space-time), we require  $\partial z = \bar{\partial} \bar{z} = 1$  and  $\partial \bar{z} = \bar{\partial} z = 0$ . The notation  $d^{-1}\Omega$  means any potential of the 3-form  $\Omega$ . In contrast with the compact version, where  $k$  is topologically quantized, the coupling  $k$  here need not be an integer, as  $H^3$  is diffeomorphic to  $\mathbb{R}^3$ , which has trivial cohomology. Another difference is that the transformation group  $SU(2)_L \times SU(2)_R$  acts on  $SU(2)$  independently on the left and right, whereas  $g \in SL(2, \mathbb{C})$  acts on  $M$  by  $M \mapsto gMg^\dagger$ , with the right factor being the Hermitian conjugate of the left one. Thus the left and right actions are tied to each other. This distinction disappears at the level of infinitesimal symmetries, since we can always pass with impunity to the complexified tangent space, thereby making the left and right actions independent.

Noncompact WZW models of the above type play a role in the functional integral version of the coset construction [18]. They and related models have been studied by high-energy physicists [52] to a certain extent, but are notoriously hard to solve completely, because of the complications that come from having to deal with the representation theory of a noncompact group. Nevertheless, they do exist as theories with a stable functional integral, and this is all we are require for now. It is reasonable to expect the above Lagrangian to be a building block of the supersymmetric theory we wish construct.

The next question is: can we combine  $H^3$  and  $S^3$  into a supersymmetric target space, giving rise to a WZW-type field theory with the properties listed in Section 5? Again, the answer will turn out to be yes. The present paper was triggered by the appearance [28, 29] of two superstring-related articles revolving around  $AdS_3 \times S^3$ , where  $AdS_3$  denotes three-dimensional anti-deSitter space. For our purposes,  $AdS_3$  is identified with the manifold of the group  $SU(1,1)$ , which preserves a quadratic form of Lorentzian signature  $(++-)$ , and has one compact and two noncompact directions. Thus the natural geometry of  $SU(1,1)$  is non-Riemannian. It is not hard to see that the product of  $SU(1,1)$  with  $SU(2) \simeq S^3$  forms the bosonic subgroup of a Lie supergroup  $PSU(1,1|2)$ , obtained from  $U(1,1|2)$  by requiring unit superdeterminant and factoring out the  $U(1)$  generated by the unit matrix. The important message from [28, 29] is that the nonlinear sigma model with target space  $PSU(1,1|2)$  is conformal for *any value of the coupling*. (In fact, this message already follows from the work of Gade and Wegner [53].) The last feature looks very attractive from our perspective, as it suggests enough flexibility to accommodate the strange critical exponents that surfaced in recent numerical work [26].

Needless to say, the  $PSU(1,1|2)$  nonlinear sigma model is *not* the field theory we want. Requirement 1 of our list says that target spaces with a non-Riemannian metric are not acceptable. Coming from statistical physics, we insist on having a field theory with an action functional bounded from below. The natural remedy, of course, is to trade  $PSU(1,1|2)$  for a related space with the required Euclidian, or definite, signature. Such a variant, with base  $H^3 \times S^3$ , was mentioned in [28], but was quickly dismissed with the statement that “there is apparently no change of variables that makes the resulting couplings to fermions real”. I fully agree with the statement that there are problems with reality concerning the fermions, and these are inevitable. Indeed, the couplings would be real if the target space arose as the stable set of some antilinear involutory automorphism of the complexified group  $PSL(2|2)$ . For the case of  $PSU(1,1|2)$ , such an automorphism is easily constructed, by adaptation from the definition of  $U(1,1|2)$  (see Section 3). However, if the stable set is to be based on the manifold  $H^3 \times S^3$ , one can show that no such automorphism exists.<sup>1</sup>

The point to be stressed here is that the superspace we are after, namely a variant of  $PSU(1,1|2)$  with Euclidian signature, is not a supermanifold in the usual sense: it exists neither as a real-analytic nor as a complex-analytic

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<sup>1</sup>Let me digress and mention in passing that the field of disordered electron systems began to battle with non-reality of fermions in 1992, when Gade [54] discovered new universality classes in sublattice models with symmetries that are now called “chiral”. It was initially thought that no supersymmetric field-theory representation of such systems exists. The solution to the puzzle was contained in a paper by Andreev, Simons, and Taniguchi [55] who noted that reality of the fermions, or of the couplings to fermions, was dispensable from a statistical physics point of view. This idea was pursued in [19] where complex conjugation of fermions was abandoned altogether, and the notion of Riemannian symmetric superspace was formulated.

supermanifold. (A supermanifold is called  $\mathbb{K}$ -analytic if the transition functions between its coordinate charts are  $\mathbb{K}$ -analytic.) Rather, it belongs to a category of objects called *Riemannian symmetric superspaces* in [19]. The distinct nature of these and more general objects has also been noted in recent lectures on supersymmetry by J. Bernstein [20]. He has drawn attention to a large category of superspaces which he calls *cs-manifolds* (“c” for complex, and “s” for super). Since the notions of Riemannian symmetric superspace and/or cs-manifold may be unfamiliar, we will now elaborate somewhat, not digging into the foundations of the subject, but reviewing the main idea and illustrating it with a well-chosen example.

According to [19], a Riemannian symmetric superspace is a highly structured object consisting of the following data. First of all, we are given a Lie supergroup  $G_{\mathbb{C}}$  and a subgroup  $H_{\mathbb{C}}$ . Both  $G_{\mathbb{C}}$  and  $H_{\mathbb{C}}$  are *complex*, so their quotient  $G_{\mathbb{C}}/H_{\mathbb{C}}$  is naturally a complex-analytic supermanifold. Secondly, by fixing a non-degenerate invariant quadratic form on  $\text{Lie}(G_{\mathbb{C}})$  (we assume such a form exists), the homogeneous space  $G_{\mathbb{C}}/H_{\mathbb{C}}$  is equipped with a  $G_{\mathbb{C}}$ -invariant supersymmetric second-rank tensor,  $\kappa$ . We say that  $G_{\mathbb{C}}/H_{\mathbb{C}}$  carries a “supergeometry”. Thirdly, the supermanifold  $G_{\mathbb{C}}/H_{\mathbb{C}}$  is based on a complex manifold  $(G_{\mathbb{C}}/H_{\mathbb{C}})_0$ . From it we select a *real* submanifold,  $M$ , with half the dimension of  $(G_{\mathbb{C}}/H_{\mathbb{C}})_0$ .  $M$  inherits a geometry from the supergeometry of  $G_{\mathbb{C}}/H_{\mathbb{C}}$  by restriction. With these provisions, we call the triple  $(G_{\mathbb{C}}/H_{\mathbb{C}}, \kappa, M)$  a Riemannian symmetric superspace (or symmetric superspace for short) if  $M$  is a Riemannian symmetric space in the classical sense [56].

The merit of the above construction is that it readily produces “nice” superspaces, while avoiding any operation of complex conjugation or adjoint for the fermions. We mention in passing that there exist ten large families of symmetric superspaces, and all of these arise in the study [19] of random matrix statistics using Efetov’s method.

The simplest nontrivial example derives from  $\text{GL}(1|1)$ , the Lie supergroup of invertible  $2 \times 2$  supermatrices  $g$ ,

$$g = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}. \quad (4)$$

We may regard  $\text{GL}(1|1)$  as a homogeneous space  $G_{\mathbb{C}}/H_{\mathbb{C}}$  constructed by quotienting  $G_{\mathbb{C}} = \text{GL}(1|1)_{\text{L}} \times \text{GL}(1|1)_{\text{R}}$  by its diagonal  $H_{\mathbb{C}} = \text{GL}(1|1)$ . The natural geometry on  $\text{GL}(1|1)$  is given by  $\kappa = \text{STr}(g^{-1}dg)^2$ . For the real submanifold  $M$  of the base  $\text{GL}(1, \mathbb{C}) \times \text{GL}(1, \mathbb{C})$  we take  $\mathbb{R}^+ \times \text{S}^1$ , the abelian group of matrices

$$g = \begin{pmatrix} e^x & 0 \\ 0 & e^{iy} \end{pmatrix}.$$

We will use the suggestive notation  $\text{H}^1$  for  $\mathbb{R}^+$ . The restriction of  $\kappa$  to  $\text{H}^1 \times \text{S}^1$  is  $dx^2 + dy^2$ , which has the attractive feature of being Riemann (or, rather,

Euclidian). Thus  $H^1 \times S^1$  is a Euclidian space in the geometry inherited from supergeometry, and the triple  $(GL(1|1), STr(g^{-1}dg)^2, H^1 \times S^1)$  belongs to our general category of Riemannian symmetric superspaces. This example indicates a general feature, namely that the real manifold  $M$  of a symmetric superspace is a product of spaces,  $M = M_B \times M_F$ , with  $M_F$  being compact and  $M_B$  noncompact. (One of the factors may of course be trivial.)

An important fact about Riemannian symmetric superspaces is that they admit an invariant Berezin integral (or superintegral) with nice properties. A brief account is as follows. The “good” integrands are (super-)functions on  $G_{\mathbb{C}}/H_{\mathbb{C}}$  that are holomorphic in a neighborhood of  $M$ . Let  $f$  denote such a function. By the principles of supergeometry, the complex-analytic supermanifold  $G_{\mathbb{C}}/H_{\mathbb{C}}$  comes with a holomorphic Berezin form,  $\omega$ , which is a rule for converting  $f$  into a holomorphic top-form  $\omega[f]$  on  $(G_{\mathbb{C}}/H_{\mathbb{C}})_0$ . The latter can be integrated over  $M$  in the usual sense, to produce the number  $\int_M \omega[f]$ . Thus, the Berezin integral is defined to be the two-step process

$$f \mapsto \omega[f] \mapsto \int_M \omega[f] .$$

The first step, called the Fermi integral or “integration over the Grassmann variables”, is benign and needs no notion of complex conjugation or reality of the fermions. Complex conjugation enters only in the second step (after the fermions have been integrated out), in order to fix the Riemannian manifold  $M$ . Thus, reality of the fermions, or even reality of the couplings to fermions, is not an issue here. Put in stronger terms, reality of the fermions is an ill-conceived and redundant concept which might as well be abolished, from our statistical physics perspective.<sup>2</sup>

We turn again to our simple  $GL(1|1)$  example for illustration. If the elements of  $GL(1|1)$  are written as in (4), the invariant holomorphic Berezin form is easily computed to be  $\omega \equiv Dg = i^{-1} da \wedge db \partial^2 / \partial \alpha \partial \beta$ , from the invariant supergeometry given by  $STr(g^{-1}dg)^2$ . The choice of normalization constant  $i^{-1}$  is a matter of convention. A quick check on the expression for  $Dg$  is provided by invariance under scale transformations: since  $d(sa) = s da$  and  $\partial / \partial (s\alpha) = s^{-1} \partial / \partial \alpha$ , the factor  $s$  drops out as required. One might now think that the volume integral  $\int Dg$  vanishes, as the fermionic derivatives  $\partial^2 / \partial \alpha \partial \beta$  have nothing to act on. If this were really so, we would have a normalization problem to solve. In the field-theoretic setting of a nonlinear sigma model with target space  $\mathbf{X} \equiv (GL(1|1), \kappa, H^1 \times S^1)$ , there would be a problem with the zero modes of the theory. However, the conclusion  $\int Dg \stackrel{?}{=} 0$  is premature and, in fact, untenable. Because the first factor in  $H^1 \times S^1$  is noncompact, we are well advised to insert

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<sup>2</sup>For this reason, I have banned the use of real-form supergroups such as  $U(1, 1|2)$  from all of my recent work; cf. the remark made in the second paragraph of Section 3.



a regularization factor and define the volume by

$$\text{vol}(\mathbf{X}) = \lim_{\epsilon \rightarrow 0} \int_{\mathbf{H}^1 \times \mathbf{S}^1} Dg \exp(-\epsilon \text{STr}(g + g^{-1})) .$$

Note that, since there are two roads to infinity on  $\mathbf{H}^1$  ( $a \rightarrow +\infty$  and  $a \rightarrow 0$ ), a factor  $e^{-\epsilon \text{STr} g}$  will not do for regularization and we really need  $e^{-\epsilon \text{STr}(g+g^{-1})}$ . An easy calculation using  $Dg = i^{-1} da \wedge db \partial^2 / \partial \alpha \partial \beta$  shows that the limit  $\epsilon \rightarrow 0$  exists, and

$$\text{vol}(\mathbf{X}) = 2\pi .$$

On the other hand, we could have considered the compact Lie supergroup  $\mathbf{U}(1|1)$  or, more generally,  $\mathbf{U}(m|n)$ . In Berezin's book [57] one can find a demonstration that the Berezin-Haar integral over all these groups vanishes identically:

$$\text{vol}(\mathbf{U}(m|n)) = \int_{\mathbf{U}(m|n)} Dg = 0 .$$

This is true in particular for  $\mathbf{U}(1|1)$ , in which case the statement can be verified by elementary means. Furthermore, since  $\mathbf{U}(m|n)$  is compact, regularization has a negligible effect here. (Incidentally, by employing the advanced machinery of localization on supermanifolds [58], one can relate the vanishing of the Berezin-Haar integral to the indefinite metric of  $\mathbf{U}(m|n)$ .)

The point we are trying to score here is this: the nonzero normalization integral ( $\text{vol} \neq 0$ ), along with the Riemannian geometry of its base, distinguishes the symmetric superspace  $(\text{GL}(1|1), \text{STr}(g^{-1}dg)^2, \mathbf{H}^1 \times \mathbf{S}^1)$  in a subtle way from its mundane counterpart  $\mathbf{U}(1|1)$ . The same subtle but significant differences exist between the Lorentzian space  $\text{PSU}(1, 1|2)$  and its Riemannian variant to be constructed below.

For mathematical perspective we conclude the section with the following elaboration. It has to be admitted that there is much redundancy in our setup for symmetric superspaces. Our declared purpose is to integrate, and the integral we want is over the real manifold  $M$  plus the fermionic “fuzz” surrounding it. A concise mathematical formulation would therefore aim to eliminate the “parent” space  $G_{\mathbb{C}}/H_{\mathbb{C}}$  and put the focus on  $M$  and a (sheaf of) algebra(s) of superfunctions on  $M$ . This can be done. Since no operation of complex conjugation of the fermions is available (and yet the “bosons”, *i.e.*, the coordinates of  $M$ , are real), the object that arises does not exist as a real-analytic or complex-analytic supermanifold. Rather, what one obtains is a *cs-manifold* in the terminology of Bernstein [20]. The local model for such a manifold is a graded commutative algebra of functions on  $U \subset \mathbb{R}^p$  with values in a complex Grassmann algebra  $\Lambda \mathbb{C}^q$ . As usual, the manifold is assembled by gluing together the local charts by means of transition functions.

To illustrate, we return to our simple example. We will give another description of the symmetric superspace  $\mathbf{X} = (\mathrm{GL}(1|1), \mathrm{STr}(g^{-1}dg)^2, \mathbb{H}^1 \times \mathbb{S}^1)$ , now reduced to a cs-manifold. For that purpose, let us introduce a local coordinate system (or “superdomain”) by the Cayley map:

$$g = \frac{1+X}{1-X}, \quad X = \begin{pmatrix} x & \xi \\ \eta & iy \end{pmatrix},$$

where  $x > 1$  parametrizes  $\mathbb{R}^+ \simeq \mathbb{H}^1$ , and  $y \in \mathbb{R}$  is a local coordinate for the circle  $\mathbb{S}^1$ . Since  $\mathbb{S}^1$  is diffeomorphic not to  $\mathbb{R}$ , but to  $\mathbb{R}$  with a point added at infinity, we need a second superdomain for an atlas of the manifold. This is constructed by setting

$$g = \sigma_3 \frac{1+X'}{1-X'}, \quad X' = \begin{pmatrix} x' & \xi' \\ \eta' & iy' \end{pmatrix},$$

with  $\sigma_3 = \mathrm{diag}(1, -1)$ . The transition functions connecting the two superdomains follow from the equations

$$\frac{1+X}{1-X} = g = \sigma_3 \frac{1+X'}{1-X'}.$$

After a little algebra, one finds

$$\begin{aligned} x' &= x - \xi\eta/iy, & \xi' &= \xi/iy, \\ \eta' &= -\eta/iy, & y' &= -1/y. \end{aligned}$$

Simply put, the cs-manifold  $\mathbf{X}$  can now be described as an algebra of “nice” functions of  $x, y, \xi, \eta$  (and, on switching superdomains, of  $x', y', \xi', \eta'$ ). Note the peculiar feature that the transition functions are complex-valued, in spite of the fact that the even coordinates  $x, y$  and  $x', y'$  are designed as real variables. Moreover, there exists no globally consistent way of imposing relations (such as  $\xi = \bar{\eta}/i$ , say) to make them real (try it!). This is characteristic of a cs-manifold: although its bosons are real, we have no choice but to leave the fermions complex and work with an algebra of complex-valued functions.

The advantage of reducing the symmetric superspace to a cs-manifold is economy of the mathematical structures used. On the negative side, the dearth of intrinsic structure to a (bare) cs-manifold make specifying a supergeometry, a necessary prerequisite for our purposes, a more involved procedure. Also, harmonic analysis and some symmetry-related aspects of the field-theoretic setting, are more transparent in the Riemannian symmetric superspace than in the cs-manifold picture. We will therefore hold on to the symmetric superspace setup. The best approach is to use *both* descriptions, the symmetric superspace and the cs-manifold, and pass between them freely to adapt to the changing computational needs. This is the approach we adopt in the sequel.

## 7 The target manifold

After these preparations, we are ready to get to the point and define the target space of the conformal field theory to be constructed. We will describe the target space first as a symmetric superspace and then as a cs-manifold.

Our starting point is  $GL(2|2)$ , the complex Lie supergroup of invertible  $4 \times 4$  supermatrices, denoted by  $g$ . This space is too large for our purposes. (The nonlinear sigma model for it generically has a critical density of states, in violation of requirement 7 of our list.) Imposing the condition of unit superdeterminant,  $SDet(g) = 1$ , reduces the complex dimension by one. The resulting supergroup,  $SL(2|2)$ , is not semisimple, since its Lie algebra still contains an abelian ideal consisting of the multiples of the unit matrix. To gain semisimplicity, we pass to the factor group  $PSL(2|2)$ , obtained by identifying in  $SL(2|2)$  the matrices that differ only by a scalar factor. The elements of  $PSL(2|2)$ , which are no longer matrices but are *equivalence classes* of matrices, are still denoted by  $g$ . We may view  $PSL(2|2)$  as a homogeneous space  $G_{\mathbb{C}}/H_{\mathbb{C}}$ , by setting  $G_{\mathbb{C}} = PSL(2|2)_L \times PSL(2|2)_R$ . The projection  $G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}/H_{\mathbb{C}}$  is done via the map  $(g_L, g_R) \mapsto g_L g_R^{-1}$ , which divides out the diagonal  $H_{\mathbb{C}} = PSL(2|2)$  ( $g_L = g_R$ ).

The Lie algebra (not the Lie superalgebra) of  $PSL(2|2)$  is a complex superspace,  $\mathfrak{psl}(2|2)$ , of dimension  $(6, 8)$ , which is to say there are 6 complex bosons and 8 complex fermions.<sup>3</sup> Supergeometry is introduced by fixing on  $\mathfrak{psl}(2|2)$  a supersymmetric quadratic form which is both nondegenerate and invariant under the adjoint action of the group. The Killing form  $\langle X, Y \rangle = \text{STr ad}(X)\text{ad}(Y)$  will not do for that purpose, as it is degenerate in the present case [59]. However, we can start from the Lie algebra of  $SL(2|2)$  in the fundamental matrix representation and take

$$\langle X, Y \rangle = \text{STr } XY .$$

This descends to a well-defined quadratic form on  $\mathfrak{psl}(2|2)$ , since

$$\langle X + s \cdot 1_4, Y + t \cdot 1_4 \rangle = \langle X, Y \rangle$$

is independent of the scalars  $s$  and  $t$ . We denote the resulting quadratic form still by  $\langle X, Y \rangle$ . This form enjoys the properties of nondegeneracy and invariance under the adjoint action of  $PSL(2|2)$ .

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<sup>3</sup>The general theory [57] instructs us to distinguish between a Lie superalgebra, and an associated Lie algebra with Grassmann structure. The former is a  $\mathbb{Z}_2$ -graded linear space  $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1$  with a superbracket or supercommutator defined on it.  $\mathcal{G}$  will usually be realized by real or complex matrices, and there are no Grassmann variables involved. The latter, the Lie algebra with Grassmann structure, is obtained by picking a large enough Grassmann algebra  $\Lambda = \Lambda_0 + \Lambda_1$  and taking the even part of the tensor product:  $\mathcal{G}(\Lambda) = \Lambda_0 \otimes \mathcal{G}_0 + \Lambda_1 \otimes \mathcal{G}_1$ . If  $\mathcal{G}$  is realized by matrices, the elements of  $\mathcal{G}(\Lambda)$  are supermatrices, with commuting entries on the even blocks, and anticommuting entries on the odd blocks. Depending on the context, the generators of  $\Lambda$  are either viewed as *parameters*, or take the role of odd *coordinates*. For present purposes, we denote the Lie superalgebra by  $\mathfrak{psl}(2, 2)$ , and the Lie algebra with Grassmann structure by  $\mathfrak{psl}(2|2)$ .

By virtue of  $\mathrm{PSL}(2|2)$  being a homogeneous space, the quadratic form  $\langle \bullet, \bullet \rangle$  induces an invariant second-rank tensor (or “metric”)  $\kappa$  on  $\mathrm{PSL}(2|2)$  by left translation. Given two tangent vectors at  $g$ , we simply move them to the Lie algebra  $\mathfrak{psl}(2|2)$  by parallel translation with  $g^{-1}$ , and then evaluate their inner product using  $\langle \bullet, \bullet \rangle$ . We denote the resulting tensor by

$$\kappa = \langle g^{-1}dg, g^{-1}dg \rangle .$$

By setting the fermionic degrees of freedom of  $\mathrm{PSL}(2|2)$  to zero, we obtain the six-dimensional complex manifold  $\mathrm{PSL}(2|2)_0$ . If  $\mathbb{C}^\times$  denotes the group of invertible complex numbers,  $\mathrm{PSL}(2|2)_0$  consists of equivalence classes,

$$\mathbb{C}^\times \mathrm{diag}(A, B) \subset \mathrm{PSL}(2|2) ,$$

parametrized by complex  $2 \times 2$  matrices  $A$  and  $B$ , which are chosen to satisfy

$$\mathrm{Det}(A) = \mathrm{Det}(B) = 1 ,$$

and hence lie in  $\mathrm{SL}(2, \mathbb{C})$ . The construction of the Riemannian symmetric superspace is completed by specifying a *real* submanifold  $M$  of  $\mathrm{PSL}(2|2)_0$ . In keeping with the discussion of Section 6, we choose  $M \simeq M_B \times M_F \simeq \mathrm{H}^3 \times \mathrm{S}^3$ . In precise terms, we take the elements of  $M$  to be of the form  $\mathbb{C}^\times \mathrm{diag}(A, B)$  where  $A$  runs through the positive Hermitian  $2 \times 2$  matrices of unit determinant:  $A = hh^\dagger$  where  $h \in \mathrm{SL}(2, \mathbb{C})$ , and  $B$  runs through  $\mathrm{SU}(2)$ . The former set ( $M_B$ ) is isomorphic to  $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2) \simeq \mathrm{H}^3$ , and the latter to  $\mathrm{S}^3$ .

The elements of the tangent space of  $M$  at the group identity are represented by pairs  $\mathrm{diag}(a, b)$  with  $a$  Hermitian and  $b$  skew-Hermitian. Hence we have

$$\langle \mathrm{diag}(a, b), \mathrm{diag}(a, b) \rangle = \mathrm{Tr} a^2 - \mathrm{Tr} b^2 = \mathrm{Tr} a^\dagger a + \mathrm{Tr} b^\dagger b \geq 0 ,$$

and  $M \simeq \mathrm{H}^3 \times \mathrm{S}^3$  is Riemann in the geometry inherited from the supergeometry of  $\mathrm{PSL}(2|2)$ . Thus  $M$  is a Riemannian symmetric space, and the triple  $(\mathrm{PSL}(2|2), \kappa, M)$  is a Riemannian symmetric superspace. Adopting standard terminology [59], we call it type  $A_1|A_1$ ; in symbols:  $\mathbf{X}_{A_1|A_1}$ .

As was discussed in the closing paragraph of Section 6, the definition of a Riemannian symmetric superspace carries some redundancy. If our purpose is to define a nonlinear sigma model, we only need the Riemannian manifold  $M \simeq \mathrm{H}^3 \times \mathrm{S}^3$  and the fermionic fuzz surrounding it, together with the invariant supergeometry induced from  $\mathrm{PSL}(2|2)$ . Hence, a more concise description of the object at hand exploits the notion of cs-manifold, which is our next topic. We can describe the cs-manifold from the coordinate point of view, by constructing an atlas of superdomains and their transition functions. Since  $\mathrm{H}^3$  is diffeomorphic to  $\mathbb{R}^3$ , and the second factor of  $M$  is a (three-)sphere, the minimal atlas consists of two superdomains. We define the first of these by using the exponential map

centered around the group unit of  $\text{PSL}(2|2)$ . In detail, we proceed as follows. We set

$$X = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \begin{pmatrix} a_3 & a_1 - ia_2 & \alpha_{11} & \alpha_{12} \\ a_1 + ia_2 & -a_3 & \alpha_{21} & \alpha_{22} \\ \beta_{11} & \beta_{12} & ib_3 & ib_1 + b_2 \\ \beta_{21} & \beta_{22} & -ib_1 - b_2 & -ib_3 \end{pmatrix},$$

and take the even coordinates  $a_1, a_2, a_3, b_1, b_2, b_3$  to be real. Note that *no conditions of any kind* are imposed on the odd coordinates  $\alpha_{ij}$  and  $\beta_{ij}$  ( $i, j = 1, 2$ ). We then exponentiate and write

$$g = \exp(X + s \cdot 1_4) \in \text{PSL}(2|2).$$

(As before,  $s$  is an arbitrary scalar.) When the odd variables are set to zero, the image of  $X + s \cdot 1_4$  under  $\exp$  lies in  $M \simeq M_B \times M_F$ , since the exponential of a traceless Hermitian matrix is a positive Hermitian matrix of determinant one, and the exponential of a traceless skew-Hermitian matrix lies in  $\text{SU}(2)$ . When the odd variables are included, exponentiation gives one superdomain of a cs-manifold with base  $M$ . A second superdomain is defined by repeating the same procedure with the group unit replaced by the equivalence class of  $\text{diag}(1_2, -1_2)$  where  $-1_2$  is the “antipode” of unity on  $\text{SU}(2) \simeq \text{S}^3$ . The element playing the role of  $X$  is now denoted by  $Y$ . To compute the transition functions connecting  $X$  with  $Y$ , we solve the equation

$$\exp(X + s \cdot 1_4) = \text{diag}(1_2, -1_2) \exp(Y + t \cdot 1_4).$$

It is not hard to see that, given the matrix entries of  $X$ , this equation on overlapping domains has a unique solution for the matrix entries of  $Y$ , and vice versa. The superdomains of  $X$  and  $Y$ , together with the transition functions relating them, constitute a cs-manifold which is the “backbone” of the Riemannian symmetric superspace  $A_1|A_1$ . This concludes our description of the target space.

## 8 Invariant Berezin integral

For our purposes, an important structure on the symmetric superspace  $A_1|A_1$  is its invariant Berezin integral. Using the cs-manifold picture, this is described as follows. It will be sufficient to use just a single superdomain, say the one centered around the group unit, which is given by  $g = \exp(X + s \cdot 1_4)$ . Let  $Dg$  denote the invariant Berezin form on  $\mathbf{X}_{A_1|A_1}$ , normalized so that it agrees with the flat Berezin form

$$DX = \prod_{i=1}^3 da_i db_i \prod_{j,k=1}^2 \frac{\partial^2}{\partial \alpha_{jk} \partial \beta_{jk}}$$

at the group unit. From the theory of Lie supergroups, we have a standard formula [57] for the Berezinian, or superjacobian,  $\mathcal{B}(X)$  of the exponential map  $X \mapsto g = \exp(X + s \cdot 1_4)$ . It reads

$$\mathcal{B}(X) = \text{SDet} \left( \frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)} \right) \Big|_{\text{psl}(2,2)} ,$$

where  $\text{ad}(X)$  acts on  $Z \in \text{psl}(2,2)$  by the commutator:  $\text{ad}(X)Z \equiv [X, Z]$ , and  $(1 - e^{-\text{ad}(X)}) / \text{ad}(X)$  is understood by its Taylor series  $\sum_{n=0}^{\infty} \text{ad}^n(-X)/(n+1)!$ . We will shortly see that the exponential map has a domain of injectivity,  $\mathcal{D}$ , delineated by

$$\sqrt{b_1^2 + b_2^2 + b_3^2} < \pi ,$$

while  $a_1$ ,  $a_2$  and  $a_3$  are unconstrained. Anticipating this fact, we express the invariant Berezin integral in the coordinates  $X$  by

$$\int_M Dg f(g) = \int_{\mathcal{D}} DX \mathcal{B}(X) f(\exp(X + s \cdot 1_4)) . \quad (5)$$

If  $f$  vanishes fast enough at infinity on  $H^3$ , this expression is correct as it stands, and there is no need to add any boundary distributions (which often appear in coordinate expressions for superintegrals) to the right-hand side.

To establish the last claim, we make the Berezinian  $\mathcal{B}(X)$  more explicit by introducing a Cartan decomposition. Let  $K \equiv \text{PSU}(2|2)$  be a *compact* real form of the complex Lie supergroup  $\text{PSL}(2|2)$ . Its elements  $k$  act on  $\text{psl}(2|2)$  by conjugation:

$$X \mapsto kXk^{-1} \equiv \text{Ad}(k)X .$$

Utilizing this action we make a polar decomposition

$$X = \text{Ad}(k)H ,$$

where

$$H = \text{diag}(x, -x, iy, -iy)$$

is diagonal, and the “radial” coordinates  $x$  and  $y$  have real range. (For this to work, we view the group  $K$  as a cs-manifold, *i.e.*, we take its fermions to be complex.) The Berezinian is a radial function,

$$\mathcal{B}(X) = \mathcal{B}(\text{Ad}(k)H) = \text{SDet} \text{Ad}(k) \Big|_{\text{psl}(2,2)} \times \mathcal{B}(H) = \mathcal{B}(H) ,$$

by the multiplicativity of  $\text{SDet}$  and the unimodularity of  $\text{PSL}(2|2)$ . An explicit formula for  $\mathcal{B}(H)$  is obtained by finding the eigenvalues, or roots, of the adjoint action  $\text{ad}(H)$  on  $\text{psl}(2,2)$ . The roots are called bosonic or fermionic depending on whether the corresponding eigenvector, or root vector, is an even or odd element of  $\text{psl}(2,2)$ . Two of the bosonic roots are zero, and the nonzero ones are  $\pm 2x$

and  $\pm 2iy$  with multiplicity one. The fermionic roots are  $\pm(x \pm iy)$ , and they have multiplicity *two*. The latter is an exceptional feature possible only in the superworld: it is known that the roots of semisimple Lie algebras (as opposed to semisimple Lie *superalgebras*) always have multiplicity one. For notational convenience in the following formula, we adopt the convention of giving the multiplicity a sign which is negative for fermionic roots (and positive for the bosonic ones). Then, if  $\Delta^+$  is a system of positive roots, the Berezinian can be written in the form

$$\mathcal{B}(H) = \prod_{\alpha \in \Delta^+} \left( \frac{\sinh \frac{1}{2}\alpha(H)}{\frac{1}{2}\alpha(H)} \right)^{2m_\alpha}.$$

The existence of the bosonic roots  $\pm 2iy$  makes  $\mathcal{B}(H)$  vanish for  $y \in \pi\mathbb{Z} \setminus \{0\}$ . Hence the injectivity domain of the exponential map is given by  $|y| < \pi$ . This translates into the condition  $\sqrt{b_1^2 + b_2^2 + b_3^2} < \pi$  stated earlier. Moreover, the singularities of  $\mathcal{B}(H)$  caused by the vanishing of the fermionic roots are located at  $x = 0$ ,  $y \in 2\pi\mathbb{Z} \setminus \{0\}$ . Therefore  $\mathcal{B}(H)$  is analytic and regular for  $x \in \mathbb{R}$  and  $-\pi < y < \pi$ . By extension,  $\mathcal{B}(X)$  is analytic and regular on the injectivity domain  $\mathcal{D}$ , and by the vanishing of  $\mathcal{B}(X)$  on the boundary  $\partial\mathcal{D}$ , no boundary distributions can occur in the coordinate expression (5) of the invariant Berezin integral.

We now turn to a property of the Berezin integral which is a prerequisite for understanding the field theory partition function. Consider the normalization integral  $\int_{\mathcal{D}} DX \mathcal{B}(X)$ . As it stands, this makes no sense, for the presence of the hyperbolic sector  $H^3$  renders the integration domain  $\mathcal{D}$  infinite. To fix the problem, we insert some convergence factor  $e^{-\epsilon h(\exp(X+s \cdot 1_4))}$ , with  $\epsilon$  small, and define the normalization integral by

$$N(\epsilon) = \int_{\mathcal{D}} DX \mathcal{B}(X) e^{-\epsilon h(\exp(X+s \cdot 1_4))}.$$

The question now is how to choose the function  $h$ . We want  $h$  to be effective as a regulator but, at the same time, it should preserve as many symmetries as possible. This leads us to choose  $h$  as a radial function:  $h(g) = h(\text{Ad}(k) \exp(H + s \cdot 1_4)) = h(\exp(H + s \cdot 1_4))$ . The first guess is to take

$$\begin{aligned} \text{STr Ad}(g) &= \text{STr } e^{\text{ad}(H)} \\ &= 1 + 1 + e^{2x} + e^{-2x} + e^{2iy} + e^{-2iy} \\ &\quad - 2e^{x+iy} - 2e^{x-iy} - 2e^{-x+iy} - 2e^{-x-iy} \\ &= -2 + 4(\cosh x - \cos y)^2. \end{aligned}$$

Near  $x = y = 0$  this varies as a fourth power  $(x^2 + y^2)^2$ , which has the undesirable feature that the Hessian degenerates to zero at that point. (This degeneracy

is an immediate consequence of the vanishing of the Killing form on  $\mathfrak{psl}(2, 2)$ :  $\text{STr ad}^2(X) = 0$ .) A better choice is

$$h(g) = \sqrt{2 + \text{STr Ad}(g)} = 2(\cosh x - \cos y) .$$

The right-hand side of this equation shows that the square root exists as an *analytic* function on all of the cs-manifold  $\mathbf{X}_{A_1|A_1}$ . For future use, note that the Taylor expansion of  $h$  around the identity coset reads

$$h(\exp(X + s \cdot 1_4)) = \frac{1}{2}\langle X, X \rangle + \dots .$$

For the choice of regulator  $e^{-\epsilon h}$  made, we now evaluate the normalization integral  $N(\epsilon)$ . This can be done by a localization principle called the Parisi-Sourlas-Efetov-Wegner theorem in disordered electron physics. Its most recent version has been stated and proved by Schwarz and Zaboronsky [58].

In the specific setting at hand, the localization principle is formulated as follows. Let  $\mathbf{X}$  be a Riemannian symmetric superspace with symmetry group  $G$  and invariant Berezin form  $D\mathbf{x}$ . Pick an odd generator  $\mathcal{F}$  of  $G$ , and denote by  $\Xi$  the Killing vector field representing the action of  $\mathcal{F}$  on  $\mathbf{X}$ . Then, if  $f$  is a function invariant under  $\Xi$  (*i.e.*,  $\Xi f = 0$ ), the integral  $\int D\mathbf{x} f$  *localizes on the zero locus* of the vector field  $\Xi$  (viewed as a differential operator). This means that the integral is determined by the values of the integrand, and a finite number of its derivatives, at the zero locus of  $\Xi$ . If  $\Xi$  is expressed in even and odd local coordinates  $x^1, \dots, x^p$  and  $\xi^1, \dots, \xi^q$  by  $\Xi = a^i(x, \xi)\partial/\partial\xi^i + \alpha^i(x, \xi)\partial/\partial x^i$ , the zero locus of  $\Xi$  is defined as the set of solutions of the equations  $a^i(x, 0) = 0$  ( $i = 1, \dots, q$ ).

The mechanism behind this version of the localization principle can be stated in a few sentences. Let the zero locus of  $\Xi$  be denoted by  $R_\Xi$  and its complement by  $\mathcal{C}$ . On the latter, the supergroup generated by  $\mathcal{F}$  acts freely, or without fixed points. This allows us to introduce local coordinates on  $\mathcal{C}$  such that the Killing vector field  $\Xi$  takes the simple form  $\Xi = \partial/\partial\xi^1$ . Now, by  $\Xi$ -invariance neither the function  $f$  nor the Berezin form  $D\mathbf{x}$  carries any dependence on  $\xi^1$ . Doing the Fermi integral over  $\xi^1$  therefore yields zero. This reasoning breaks down on the set  $R_\Xi$ , but it remains valid outside an arbitrarily small neighborhood of  $R_\Xi$ . As a result, the integral  $\int D\mathbf{x} f$  can only depend on a finite number of terms in the Taylor expansion of  $f$  at  $R_\Xi$ .

This principle allows us to compute the normalization integral  $N(\epsilon)$  rather easily. Because the regulator  $e^{-\epsilon h}$  is a function with a high degree of symmetry, there exist several odd vector fields with the required properties in order for the localization principle to take effect. It will be sufficient to use just one of them. Fix  $i, j \in \{1, 2\}$ , and let  $E_{ij}$  be the  $2 \times 2$  matrix whose entries are zero everywhere except on the intersection of the  $i$ -th row with the  $j$ -th column where the entry is unity. Put

$$\mathcal{F} = \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} ,$$



and, with  $\sigma$  an odd parameter, define  $\Xi$  by

$$(\Xi f)(g) = \frac{\partial}{\partial \sigma} f(e^{\sigma \mathcal{F}} g e^{-\sigma \mathcal{F}}) .$$

$\Xi$  is a Killing vector field since conjugation by  $e^{\sigma \mathcal{F}}$  preserves the supergeometry of  $\mathbf{X}_{A_1|A_1}$ . Moreover, any radial function  $f$  satisfies  $f(e^{\sigma \mathcal{F}} g e^{-\sigma \mathcal{F}}) = f(g)$ , and is therefore invariant w.r.t.  $\Xi$ . In particular, the regulator  $e^{-\epsilon h}$  is  $\Xi$ -invariant. Hence, by the principle stated, the normalization integral  $N(\epsilon)$  localizes on the zero locus of  $\Xi$ . We claim that the latter consists of only a single point, namely the origin  $X = 0$  of  $\mathbf{X}_{A_1|A_1}$ . To see that this is so, we examine the infinitesimal action of  $e^{\sigma \mathcal{F}}$  on the radial space, or the Cartan subalgebra. This action is determined by the root corresponding to the root vector  $\mathcal{F}$ , which is one of the set  $\pm(x \pm iy)$  and vanishes only for  $x = y = 0$ , or  $X = 0$ , as claimed. (Note that the Riemannian nature of  $\mathbf{X}_{A_1|A_1}$  is crucial for this argument. If we were working on  $\text{PSL}_{\mathbb{R}}(2|2)$  or  $\text{PSU}(2|2)$ , the roots would be  $\pm(x \pm y)$  or  $\pm i(x \pm y)$ , giving a much bigger set for the zero locus of  $\Xi$ .) The implication is that we may Taylor expand the integrand around  $X = 0$  and reduce  $N(\epsilon)$  to a Gaussian integral:

$$N(\epsilon) = \int DX \exp -\frac{\epsilon}{2} \langle X, X \rangle ,$$

which is readily calculated to be  $N(\epsilon) = \pi^3 \epsilon$ . Here we recognize the utility of defining the regulator  $h$  by taking a square root, so as to have a nondegenerate Hessian at  $X = 0$ : a vanishing Hessian would have forced us to expand to fourth order in  $X$ .

We thus see that the normalization integral  $N(\epsilon)$  is completely determined by Gaussian fluctuations around the point  $X = 0$ . The same is true for a large class of integrals on  $\mathbf{X}_{A_1|A_1}$ . Note that the symmetry group of  $\mathbf{X}_{A_1|A_1}$  provides us with a total of 8 fermionic Killing vectors. In order for localization onto  $X = 0$  to occur, it is sufficient for the integrand to be invariant under a *single one* of these vector fields. (Recall that this is a consequence of the fermionic root system being  $\pm(x \pm iy)$ .) The quantum numbers  $i, j = 1, 2$  counting these symmetries originate from the distinction between retarded and advanced Green functions of the disordered electron system. Let us put

$$\mathcal{F}_{ij}^+ = \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{F}_{ij}^- = \begin{pmatrix} 0 & 0 \\ E_{ji} & 0 \end{pmatrix} \quad (6)$$

and call the sector  $i = 1$  “retarded” and  $i = 2$  “advanced”. The correlation functions or observables which involve only retarded Green functions will be invariant under the Killing vector fields of  $\mathcal{F}_{22}^{\pm}$ , and those involving only advanced information will be invariant under the Killing vector fields of  $\mathcal{F}_{11}^{\pm}$ . As a consequence of the localization principle, such observables are trivial. We will return to this point later, in the field-theoretic setting.

We can get a more complete perspective on the Berezin integral for  $\mathbf{X}_{A_1|A_1}$  by making the polar decomposition

$$X = \text{Ad}(k)H .$$

Under this substitution, the flat Berezin form  $DX$  transforms as

$$DX \longrightarrow j(H)dH Dk .$$

Here  $Dk$  is an invariant Berezin form on  $K/T$ , where  $T \simeq \text{U}(1) \times \text{U}(1)$  is a maximal torus,  $dH = dx dy$  is a Euclidian radial measure, and  $j(H) = \text{SDet ad}(H)|_{\mathcal{T}_o(K/T)}$  is the superdeterminant of  $\text{ad}(H)$  acting on the tangent space of  $K/T$  at the origin  $o \equiv T$ . The product of the Berezinians of the two transformations  $(k, H) \mapsto X = \text{Ad}(k)H$  and  $X \mapsto g = \exp(X + s \cdot 1_4)$  is

$$J(H) = j(H)\mathcal{B}(H) = \frac{\sinh^2 x \sin^2 y}{(\cosh x - \cos y)^4} .$$

The invariant Berezin integral for  $X_{A_1|A_1}$  now takes the form

$$\int_M Dg f(g) = \mathcal{R}[f] + \int \left( \int_{K/T} Dk f(k \exp(H + s \cdot 1_4)k^{-1}) \right) J(H)dH \quad (7)$$

where the radial integral is restricted to run over a Weyl chamber  $0 < x < \infty$  and  $0 < y < \pi$ . The first term on the right-hand side is a boundary distribution, which shows up as a consequence of the singularity of the function  $J(H)$  at the origin  $x = y = 0$ . The general theory of boundary distributions for polar coordinate integrals on symmetric superspaces yields an explicit formula for  $\mathcal{R}[f]$ . We will skip the details here, as they form the subject of a separate paper [60]. In brief, the idea is to vary the “bulk term” (the second term on the right-hand side of (7)) by an infinitesimal isometry of the superspace. By partial integration on  $K/T$ , this variation can be manipulated to become the radial integral of an exact form, and application of Stokes’ formula then converts it into an integral over the boundary of the radial space. By invariance of the complete integral, this boundary term must be exactly cancelled by the variation of  $\mathcal{R}[f]$ . In this way, one obtains the result

$$\mathcal{R}[f] = (Lf)(g_0) ,$$

where  $L$  denotes the Laplace-Beltrami operator, and  $g_0$  the identity coset  $e^s \cdot 1_4$ .

The formula (7) holds for any integrable function  $f$ , radial or not. We now apply it to the normalization integral  $N(\epsilon)$ . In that case the second term on the right-hand side of (7) disappears, since  $f = e^{-\epsilon h}$  is radial, and  $\int Dk = 0$ . The last statement follows from the fact that the volume of  $K = \text{PSU}(2|2)$  is

zero, as is the volume of  $U(2|2)$ . It can also be understood from the localization principle, by noting that  $K$  acts on  $K/T$  without fixed points. Hence,

$$N(\epsilon) = \int_M Dg e^{-\epsilon h(g)} = \mathcal{R}[e^{-\epsilon h}] = (L e^{-\epsilon h})(g_0) ,$$

and by applying the Laplacian to  $e^{-\epsilon h}$  at the identity coset  $g_0$ , we recover the result  $N(\epsilon) = \pi^3 \epsilon$ .

## 9 The WZW model of type $A_1|A_1$

After this spacious presentation of mathematical background, we are going to discuss a nonlinear sigma model of maps from  $\Sigma$ , the configuration space for a single electron of the two-dimensional electron gas, into the Riemannian symmetric space  $\mathbf{X}_{A_1|A_1}$ . Although the latter is not a group, its complexification is. To remind ourselves of this fact, we denote the field by  $g$ . It is also important to keep in mind that the target space does not have a representation in  $\text{Mat}(2|2)$  but consists of *equivalence classes* of supermatrices. This places a constraint on the terms that may appear in the field-theory Lagrangian.

The principal term of the nonlinear sigma model action is given by

$$S_0 = \frac{1}{2\pi} \int_{\Sigma} d^2x \langle g^{-1} \partial g, g^{-1} \bar{\partial} g \rangle .$$

To break parity of the field theory (or parity of the disordered electron gas) we need a term of the Wess-Zumino-Novikov-Witten type:

$$\Gamma = \frac{1}{24\pi} \int_{\Sigma} d^{-1} \langle g^{-1} dg, [g^{-1} dg, g^{-1} dg] \rangle ,$$

Because of the presence of  $M_F \simeq S^3$  in the base  $M = M_B \times M_F$  of  $\mathbf{X}_{A_1|A_1}$ , single-valuedness of  $e^{-ik\Gamma}$  quantizes the coupling  $k$  to be an integer [61]. The two functionals  $S_0$  and  $\Gamma$  have zero modes (the constant fields), so we include a term proportional to

$$S_{\text{reg}} = \int_{\Sigma} d^2x h(g)$$

to regularize them. The complete action is then

$$S = f^{-2} S_0 + ik\Gamma + \epsilon S_{\text{reg}} , \tag{8}$$

where  $f$  is a coupling constant, and  $\epsilon$  is a positive infinitesimal. We call this two-parameter field theory the  $A_1|A_1$  WZW model. We use that name although the model does not have the Kac-Moody symmetry  $g(z, \bar{z}) \rightarrow \Omega(z)g(z, \bar{z})\bar{\Omega}(\bar{z})$  of a conventional WZW model unless  $f^2 = 1/|k|$ . (An alternative terminology such

as “principal chiral model” or the plain “nonlinear sigma model” would ignore the WZW term, whose presence is vital for the application to the QH plateau transition.)

Note that the theory (for  $\epsilon = 0$ ) is invariant under infinitesimal chiral transformations  $g(x) \mapsto X_L g(x) - g(x) X_R$  with  $X_L, X_R \in \mathfrak{psl}(2|2)$ . This exponentiates to an invariance under

$$g(x) \mapsto g_L g(x) g_R^{-1}$$

as long as  $g_L, g_R \in \mathrm{PSL}(2|2)$  are close enough to the group unit. It has to be said that such a transformation is not an isometry of  $\mathbf{X}_{A_1|A_1}$  for general  $g_L, g_R$ . (In particular, recall that the left and right actions of  $\mathrm{SL}(2, \mathbb{C})$  on  $H^3 \subset \mathbf{X}_{A_1|A_1}$  are related to each other by a unitarity condition.) Nevertheless, the functional integral does remain invariant, by a functional generalization of what is called Cauchy’s theorem in complex analysis. The situation is the same as for the Haar integral of a compact Lie group  $U$ :

$$\int_U F(g) dg = \int_U F(g_L g g_R^{-1}) dg ,$$

the invariance of which is not restricted to  $g_L, g_R \in U$ , but can hold (depending on the analytic properties of  $F$ ) more generally for  $g_L$  and  $g_R$  in the complexification of  $U$ .

The invariance under  $G_L \times G_R$  is the chiral symmetry which we expect to emerge for the disordered electron system at criticality, in view of our discussion of the loop-group coherent state path integral for the superspin chain in Section 4. Away from the critical point, where chiral symmetry is broken to the diagonal ( $g_L = g_R$ ), the low-energy physics should be described by Pruisken’s nonlinear sigma model. Hence, Pruisken’s model has to sit inside the WZW model, and the field  $Q$  of the former is somehow related to the field  $g$  of the latter. To describe the relation, write  $G \equiv \mathrm{PSL}(2|2)$ , and let  $H \subset G$  be the subgroup of elements  $h$  that stabilize  $\Sigma_3 = \mathrm{diag}(+1, -1, +1, -1)$  under conjugation:  $h \Sigma_3 h^{-1}$ . (This is the same  $\Sigma_3$  as in Section 3, but we have found it convenient to rearrange the basis, and thus the ordering of matrix entries, from “retarded first, advanced last” to “bosons first, fermions last”.) The quotient  $G/H$  is naturally isomorphic to  $\mathrm{GL}(2|2)/\mathrm{GL}(1|1) \times \mathrm{GL}(1|1)$ . We can implement the projection from  $G$  to  $G/H$  by mapping  $g$  onto  $g \Sigma_3 g^{-1}$ , and this descends to a projection from  $\mathbf{X}_{A_1|A_1}$  onto  $\mathrm{U}(1, 1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$ , which is Pruisken’s target space. We are thus led to put

$$Q = \mathrm{Ad}(g) \Sigma_3 = g \Sigma_3 g^{-1} . \tag{9}$$

There exists another strong motivation for making this identification. Requirement 7 of our list says that the density of states of the disordered electron system is *noncritical* at the critical point. It must therefore be represented in the field theory by an operator of vanishing scaling dimension. In Pruisken’s theory, the

density of states is known to be represented by the operator  $\text{STr}\Sigma_3 Q$ . To arrange for its scaling dimension to be zero, we should construct it from the adjoint representation of  $\text{PSL}(2|2)$ , which has vanishing quadratic Casimir. (From a one-loop computation, the scaling dimension of a local operator transforming according to a representation  $\rho$ , is proportional to the quadratic Casimir invariant evaluated on  $\rho$ . It has been conjectured [29] that in the present case this relation is not changed by higher-loop corrections.) This observation, together with the necessity to project out  $H$ , strongly suggests  $Q = \text{Ad}(g)\Sigma_3$ .

For the purpose of calculating  $\Omega(a_0, a_1; b_0, b_1)$ , the correlator of spectral determinants of Section 2, we perturb the field theory by adding an extra term

$$S_\omega = \Lambda^2 \int_\Sigma d^2x \text{STr} (\omega \text{Ad}(g)\Sigma_3 - \omega \Sigma_3)$$

where, in the ordering chosen (bosons first, fermions last),

$$\omega = \text{diag}(\ln a_0, -\ln b_0, \ln a_1, -\ln b_1) ,$$

and  $\Lambda$  is a UV cutoff. This form of  $S_\omega$  follows from the frequency term  $L_\omega$  of Pruisken's model, on setting  $Q = \text{Ad}(g)\Sigma_3$  and reordering the matrix entries.

Let us then restate the basic proposition of the present paper: the  $A_1|A_1$  WZW model as defined above (with coupling constants  $f$  and  $k$  that will be fixed in due course) is the conformal field theory describing the critical physics of electron delocalization at the transition between quantum Hall plateaus.

Our job in the following is to check the list of requirements laid down in Section 5. The first one was that the functional integral be well-defined and stable. We have taken great care to define the target space as a symmetric superspace with a Riemannian metric on its base. We have also taken care of the zero modes. I believe that nothing else needs to be said about this subject except that, of course, the theory needs regularization in the ultraviolet. It is natural to take the short-distance cutoff as the largest of the microscopic lengths of the disordered electron gas.

## 10 BRST invariance

The second requirement of our list is manifest conformal invariance of the field theory. This is not a pressing issue and is postponed until the next section. The most urgent problem to address are requirements 3 and 4, concerning the normalization of the partition function of the field theory and the triviality of some of its correlation functions. These two points are closely related (the former is a consequence of the latter) and can be dealt with in a single shot.

Consider the correlator of spectral determinants  $\Omega(a_0, a_1; b_0, b_1)$  defined in (1). For elementary reasons that were spelled out in Section 2, it reduces to

unity on setting either  $a_0 = a_1$  or  $b_0 = b_1$ . How does the field theory manage to perform the trick of reproducing this feature? The short answer is that the theory becomes *topological* and undergoes dimensional reduction, by one or several of its many supersymmetries. (The mechanism is nicely explained in [62].) To see concretely how the reduction works, we consider the case  $a_0 = a_1$ . For these parameter values, the action functional  $S + S_\omega$  acquires a global invariance under supersymmetry transformations in the retarded sector ( $i = j = 1$ ), since the first and third entry of the diagonal matrix  $\omega$  are now equal. The transformations are given by

$$\delta g(x) = [\sigma \mathcal{F}_{11}^\pm, g(x)] ,$$

where  $\sigma$  is an odd parameter, and  $\mathcal{F}_{11}^\pm$  are two of the eight fermionic generators  $\mathcal{F}_{ij}^\pm$  of  $\mathfrak{psl}(2, 2)$ , see (6). Viewing these generators as BRST operators, we may express the global invariance of  $S + S_\omega$  by saying that the theory is BRST-closed. The BRST transformations can be recast in the form

$$g^{-1} \delta g = (\text{Ad}(g) - 1) \sigma \mathcal{F}_{11}^\pm .$$

To exploit the BRST symmetry, we note that the only fixed point of the BRST transformations is the constant field configuration  $g(x) \equiv g_0 \equiv e^s \cdot 1_4$ . The reason is the same as in the zero-dimensional case:  $\text{Ad}(g)$  on any one of the fermionic root vectors  $\mathcal{F}_{ij}^\pm$  equals unity if and only if  $g$  is the *origin* of  $\mathbf{X}_{A_1|A_1}$ . (Once again, the Riemannian nature of  $\mathbf{X}_{A_1|A_1}$  is crucial here, and the situation would be less favorable if we were working on  $\text{PSL}_{\mathbb{R}}(2|2)$  or  $\text{PSU}(2|2)$ .) Being fermionic, the generators  $\mathcal{F}_{11}^\pm$  square to zero. As a result, if we *exclude* from the functional integral an infinitesimal ball  $B_\varepsilon$  surrounding the BRST fixed point  $g(x) = g_0$ , the theory becomes BRST-exact. In other words, on the complement of  $B_\varepsilon$  the functional integrand is a “total derivative” w.r.t. some odd collective coordinate  $\sigma$ . Doing the Fermi integral over  $\sigma$  yields zero.

The field theory then collapses onto  $B_\varepsilon$ , and we can compute the functional integral by simply carrying out leading-order perturbation theory around the configuration  $g(x) = g_0$ . The rest is easy. Expansion of  $S + S_\omega$  around that configuration up to quadratic order in  $X$  gives

$$S^{(2)} = \int d^2x \left( \frac{1}{2\pi f^2} \langle \partial X, \bar{\partial} X \rangle + \frac{\epsilon}{2} \langle X, X \rangle + \frac{\Lambda^2}{2} \langle \omega, \text{ad}^2(X) \Sigma_3 \rangle \right) ,$$

and by doing the Gaussian integral over  $X$  we obtain the expression

$$\text{const} \times \text{Det}(K) \frac{\text{Det}(K + \Lambda^2 \ln(a_1 b_0)) \text{Det}(K + \Lambda^2 \ln(a_0 b_1))}{\text{Det}(K + \Lambda^2 \ln(a_0 b_0)) \text{Det}(K + \Lambda^2 \ln(a_1 b_1))} , \quad (10)$$

where  $K$  is the operator  $K = -(\pi f^2)^{-1} \partial \bar{\partial} + \epsilon$ . The first factor is a constant due to the choice of normalization of the functional integral measure, which we are

still free to specify. It is understood that  $a_0 = a_1$ , as was assumed at the outset of our calculation. Then the determinants in the numerator and denominator cancel pairwise, and we are left with

$$\Omega(a, a; b_0, b_1) = \text{const} \times \text{Det}(K) .$$

Obviously, in order for this to be unity, we must choose the normalization to be

$$\text{const} = \text{Det}^{-1}(K) .$$

We can make this choice explicit by shifting the action  $S \rightarrow S + \ln \text{Det}(K)$ . Alternatively, we can introduce an auxiliary field, say a complex free boson  $\varphi$  (or, equivalently, two real free bosons) and pass to

$$S' = S + \int_{\Sigma} d^2x \bar{\varphi}(x)(K\varphi)(x) .$$

What we did for  $a_0 = a_1$  can be repeated *mutandis verbis* for  $b_0 = b_1$ . Summarizing the two cases we have

$$\Omega(a, a; b_0, b_1) = \Omega(a_0, a_1; b, b) = 1 .$$

In particular, it follows that the partition function is normalized to unity:  $Z = \Omega(1, 1; 1, 1) = 1$ .

One might now object that we are enforcing normalization by appealing to an auxiliary field  $\varphi$  the origin of which we did not explain. This objection has to be taken seriously. Normalization of the partition function is a very basic and robust feature of the supersymmetric field-theory formalism for disordered electron systems. It holds independently of all the parameters of the system (such as disorder strength, system size, lattice constant *etc.*) and in particular, applies whether the system is critical or not. And, indeed, in Pruisken's nonlinear sigma model the normalization always *is* unity. This happens by the same BRST mechanism that was utilized above, except that the bosonic and fermionic degrees of freedom are now equal in number, so all determinants cancel exactly. Thus, normalization is automatic, and no fine-tuning of the parameters is needed.

One might therefore be worried by the dependence of the normalization factor  $\text{Det}^{-1}(K)$  on the coupling  $f$  and the ultraviolet cutoff  $\Lambda$ . (The dependence on  $\epsilon$  is *not* an issue. This parameter is a positive infinitesimal whose only role is to regularize the zero modes. Alternatively, we could set  $\epsilon = 0$  and modify the field configuration space by taking the quotient by the zero modes.) Our response is this. The theory we have written down is not claimed to be applicable to the whole range of parameters of the quantum Hall system, but is specifically made for its critical point. The theory is imagined to be the endpoint of a renormalization group (RG) trajectory starting, say, from Pruisken's model at weak

coupling (and  $\sigma_{xy} = 1/2$ ). Our working hypothesis is that the boson|fermion count, which is 4|4 in Pruisken's model is promoted to 6|8 in the WZW model at the fixed point. In Pruisken's model the normalization is unity, and we can be sure that supersymmetry acts to keep it there all the way along the RG trajectory. Therefore two auxiliary real bosons must appear to absorb the change in normalization due to the mismatch 6 versus 8. From this perspective, the quantities  $f$  and  $\Lambda$  entering  $\text{Det}(K)$  are not parameters at our disposal, but are numbers determined by the RG flow.

If the theory is to make sense as an RG fixed point for the disordered electron system, what we must demand is that changing the cutoff  $\Lambda$  preserves the normalization. And, in fact, it does: the operator associated to the coupling constant  $f$  is truly marginal in the auxiliary boson theory and also in the WZW model (see the next section). Therefore, once the normalization has been set to unity, it stays there when a RG transformation is applied.

The above mechanism of dimensional reduction would not be convincing if it were limited to the partition function and the spectral correlator  $\Omega$ . Fortunately, it applies more generally, and the same reduction occurs for any observable which is already present in Pruisken's theory and is invariant under supersymmetry (or BRST) transformations in the retarded or advanced sector. The argument goes as follows. Consider any correlation function

$$\left\langle F_1(Q(x_1))F_2(Q(x_2))\dots F_p(Q(x_p)) \right\rangle, \quad (11)$$

where the  $F_i$  are local functionals of  $Q$ . Now assume BRST invariance in the retarded or advanced sector. Then, by the same reasoning as before, the functional integral will localize onto the BRST fixed point  $g(x) \equiv g_0$ . What remains to be done is the (Gaussian) integral over the small fluctuations around  $g_0$ . For this purpose we decompose the Lie algebra of  $G = \text{PSL}(2|2)$  as

$$\text{Lie}(G) = \text{Lie}(H) + \mathcal{P},$$

where  $H \subset G$  still is the stabilizer of  $\Sigma_3 \in \text{Lie}(G)$  under the adjoint action. We then put

$$g(x) = e^{Y(x)}e^{X(x)}g_0,$$

where  $X(x) \in \text{Lie}(H)$  and  $Y(x) \in \mathcal{P}$ . By design,  $Q$  depends only on  $Y$ :

$$Q(x)\Sigma_3 = e^{Y(x)}\Sigma_3 e^{-Y(x)}\Sigma_3 = e^{2Y(x)}.$$

Substituting these expressions into the functional integral and expanding around  $g_0$ , we find that the integrand separates into two factors, one depending on  $X$  and another depending on  $Y$ :

$$\langle F_1(Q(x_1))\dots F_p(Q(x_p)) \rangle = \langle 1 \rangle_X \times \langle F_1(\Sigma_3 + \dots) \dots F_p(\Sigma_3 + \dots) \rangle_Y.$$



(Although  $\text{Lie}(G)$  and  $\mathcal{P}$  are vector spaces over  $\mathbb{C}$ , it is understood from Section 6 that the bosonic degrees of freedom in  $X$  and  $Y$  are integrated only over the *real* directions specified by the tangent spaces of  $\mathbf{X}_{A_1|A_1}$  and its projection  $U(1,1|2)/U(1|1) \times U(1|1)$ .) Because  $X$  does not appear in  $Q$ , the integral over  $X$  still gives  $\text{Det}(K)$ , which is cancelled by the inverse determinant from the auxiliary complex boson. The remaining integral over  $Y$  is more complicated, but will work out to give the “right” answer. The reason is that the very same integral is obtained when evaluating the BRST-invariant correlator (11) in Pruisken’s nonlinear sigma model. In the latter case, too, the BRST mechanism reduces the functional integral to small fluctuations around  $Q(x) = \Sigma_3$ . In Gaussian approximation, the action functionals of Pruisken’s model and the WZW model formally agree (on setting  $Q = g\Sigma_3g^{-1}$ ), and to make them coincide we put  $\sigma_{xx} = (8\pi f^2)^{-1}$ . Hence the dimensionally reduced theories are the same, and we conclude

$$\left\langle F_1(Q(x_1)) \dots F_p(Q(x_p)) \right\rangle_{\text{WZW}} = \left\langle F_1(Q(x_1)) \dots F_p(Q(x_p)) \right\rangle_{\text{Pruisken}}.$$

Thus the question whether BRST-invariant correlators of the form (11) are trivial and are correctly normalized in the WZW model, reduces to the same question in Pruisken’s theory. We may take it for granted that the answer to the latter question is in the affirmative.

## 11 Conformal invariance

With the normalization issue taken care of, we now turn to requirement 2: conformal invariance. This point has already been argued convincingly in two recent papers for the target spaces  $\text{PSU}(2|2)$  and  $\text{PSL}_{\mathbb{R}}(2|2)$ , so we can afford to be brief. In Ref. [28] the action functional of the  $\text{PSU}(n|n)$  nonlinear sigma model was related to that for  $U(n|n)$ . It was then shown that the term

$$S_0 = (2\pi f^2)^{-1} \int d^2x \langle g^{-1} \partial g, g^{-1} \bar{\partial} g \rangle$$

does not renormalize in  $U(n|n)$ , and hence not in  $\text{PSU}(n|n)$ , for any value of the coupling constant  $f$ . The vanishing of the beta function for  $S_0$  had already been pointed out by Gade and Wegner in their 1991 paper [53] on  $U(m)$  at  $m = 0$ , which is perturbatively equivalent to  $U(n|n)$ . The result can be understood qualitatively from the fact that the beta function of  $S_0$  for  $U(n|n)$  is independent of  $n$ , and for  $n = 1$  the gauged theory  $\text{PSU}(1|1)$  is free and finite. In Ref. [29], essentially the same nonlinear sigma model [with target space  $\text{PSL}_{\mathbb{R}}(2|2)$ ] was considered. There, conformal invariance was attributed to the quadratic Casimir invariant being zero in the adjoint representation. The vanishing of the Casimir causes the vanishing of certain invariant current-current correlation functions,

which in turn protects the component  $T_{z\bar{z}}$  of the energy-momentum tensor (which is zero in the classical theory) from becoming nonzero as a result of quantum fluctuations. These arguments are independent of the signature of the metric of the manifold and apply equally to our case. They are also robust enough to accommodate the presence of a WZW term. Thus the important message is that we have a family of conformal field theories with *two* parameters  $f$  and  $k$  to play with. The Virasoro central charge of these theories is easily seen to be  $c = -2$ .

By standard manipulations done on the Lagrangian, the holomorphic part of the energy-momentum tensor has the classical form

$$T(z) = \frac{1}{2}f^2 \langle g^{-1} \partial g, g^{-1} \partial g \rangle + \frac{1}{2}f^2 \partial \bar{\varphi} \partial \varphi ,$$

which is independent of the topological coupling  $k$  and is not changed by quantum fluctuations. On general grounds [63], the singular part of the operator product expansion of  $T(z)$  with itself must be of the standard form

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \dots$$

The Virasoro central charge of a complex free boson (or of a pair of real free bosons) is  $+2$ . Together with the contribution from the WZW field  $g$ , this gives a total central charge of  $c = -2 + 2 = 0$  in our case. (The central charge vanishes here for the trivial reason that the field theory partition function always equals unity, independent of the size of the system. A more informative quantity can be introduced by following a recent proposal by Gurarie [64].) This is all we have to say about conformal invariance here.

## 12 More checks

We now check the next three points on our list. Of these, numbers 5 and 7 have already been input in the course of our development, so we will be brief. The former says that the fixed-point theory must have a global chiral symmetry  $\mathrm{SL}(2|2)_{\mathrm{L}} \times \mathrm{SL}(2|2)_{\mathrm{R}}$ , and the central subgroup of  $\mathrm{SL}(2|2)$  generated by the unit matrix has to be represented as a gauge degree of freedom. Both features are automatic by the construction of the target space  $\mathbf{X}_{A_1|A_1}$  and the choice of Lagrangian made. Note also that, although  $\mathrm{GL}(2|2)$  does not act on  $\mathrm{SL}(2|2)$ , it does act on  $Q = g \Sigma_3 g^{-1}$  ( $g \in \mathrm{SL}(2|2)$ ) by conjugation, so the symmetries of Pruisken's nonlinear sigma model are fully present.

Number 7 requires the density of states to be noncritical. In Pruisken's model, this observable is represented by the operator  $\mathrm{STr} \Sigma_3 Q$ . We have already identified the field  $Q$  with  $\mathrm{Ad}(g)\Sigma_3$  in the WZW theory. The field  $Q$ , and the density of states along with it, will be noncritical if its scaling dimension is zero. The latter is in fact the case here. On general symmetry grounds the

scaling dimension of an operator must be expressed by the corresponding Casimir eigenvalues, and it can be shown [29] that all Casimir invariants vanish in the adjoint representation of  $\mathfrak{psl}(2, 2)$ .

Requirement 6 concerns the fate of the theory under a perturbation

$$\text{STr } \omega (\text{Ad}(g) - 1) \Sigma_3$$

with  $\omega = \text{diag}(a_0, 1, b_0, 1)$ . As can be seen, for example from the result (10), such a perturbation gives a mass  $-\ln(a_0 b_0)$  to some fields in the Boson-Boson sector, and masses  $-\ln(a_0)$  or  $-\ln(b_0)$  to some of the fermionic fields. In the infrared limit, these degrees of freedom disappear from the theory. What is left behind is the Fermion-Fermion sector, which remains massless, being protected by a residual  $\text{SU}(2)_L \times \text{SU}(2)_R$  symmetry. Restriction of the theory to this sector yields an  $\text{SU}(2)$  WZW Lagrangian with topological coupling  $k$ . Requirement 6 then says that the integer  $k$  has to take its smallest nonzero value:

$$k = 1. \tag{12}$$

This determines one of the two coupling constants of the theory. The correct choice of the other coupling  $f$  is less obvious.

## 13 The coupling constant $f$

Renormalization group fixed points in conventional chiral nonlinear sigma models with WZW term, come as a discrete one-parameter family: once a value for the level  $k$  has been chosen, the other coupling has to be  $f^2 = 1/|k|$  in order for the theory to be conformally invariant. In the present case the situation is different. Given a value for  $k$ , conformal invariance does not determine  $f$ , which is a truly marginal coupling parametrizing a *line* of fixed points  $f^2 \leq 1/|k|$ .

While being an intriguing feature, the existence of a truly marginal perturbation is a threat to universality and therefore problematic for our proposal. Unless there exists some hidden symmetry or other constraint, it stands to reason that the marginal direction can be explored by the disordered electron gas: since the renormalization group trajectories of different members of the QH family originate from different initial conditions, we expect the critical trajectories to terminate on different points on the fixed line, leading to a variety of critical behavior. In contradistinction, the critical properties seen in numerical and real experiments on the QH transition appear to have a high degree of universality. The latter seems to suggest that the critical behavior is governed by a single fixed point, not a one-parameter family of such points.

The question, then, is how the observed universality can be reconciled with the marginality of  $f$ . What we need is some mechanism to ensure that the critical RG trajectories for different members of the QH universality class intersect the

fixed line at the same point. The scenario we offer is this. The coupling constant  $f$  determines the most singular terms (proportional to unity) of the operator product expansion for the vertex operators of the field theory. Thus the value of  $f$  controls the short-distance singularities. In particular,  $f$  controls the fluctuations of the WZW field at short distances from a conducting (or absorbing) boundary. At the same time, the short-distance physics of the disordered electron gas near a conducting boundary is *classical diffusion* of electrons. As is well understood, the conductance and other observables can be expressed as sums over electron paths. In a typical path, loops are prevalent in the bulk of the system, but rare in the vicinity of an absorbing boundary. Thus, while the dynamics in the bulk is strongly influenced by quantum interference effects (due to loops) and incipient localization, interference near a boundary is cut off by absorption (or exiting probability flux) and the motion can be treated as being classical there. (“Classical” here means that the absolute square of a sum of path amplitudes is well approximated by an incoherent sum of squares.) In the field theory, classical diffusion corresponds to small field fluctuations around a “vacuum” selected by the absorbing boundary conditions. This correspondence is, in fact, what leads to the identification, in Pruisken’s theory, of the coupling  $\sigma_{xx}$  with the classical longitudinal conductivity (which in turn is proportional to the classical diffusion constant by the Einstein relation). The same identification will be made for  $1/8\pi f^2$  in the WZW model. By this argument, the parameter  $f$  is completely determined by matching to the classical dynamics near a conducting boundary. Universality is therefore guaranteed to the extent that the classical diffusion constant has a universal value. Evidence in favor of the latter is provided by the semicircle relation

$$\sigma_{xx}^2 + (\sigma_{xy} - 1/2)^2 = 1/4 , \quad (13)$$

which has been argued to be universally valid for incoherent QH systems [65].

Our strategy in the sequel will be as follows. As was first pointed out in [45], charge transport in a critical conductor is probed least obtrusively by measuring a conductance between *interior* contacts. The simplest theoretical setup is to take an infinite system with two interior contacts that are small. At criticality, the conductance will then be some algebraically decaying two-point function of conformal field theory. The critical exponent determining the decay at large distances between the two contacts is expected to be universal, but nontrivial to compute. (If the contact in field-theoretic representation is expanded in scaling fields with scaling dimension  $\Delta_\lambda$ , the exponent equals  $2\text{Min}\Delta_\lambda$ .) On the other hand, the form of the short-distance singularity of the conductance depends on the size or “strength” of the contacts. If the contacts are strong, so that an electron near a contact quickly exits and the time spent in the system is short, the conductance becomes *classical* at small distances between the two contacts. (In contrast, for contacts that are point-like or weak, the classical transport regime does not exist, as the behavior of the conductance is controlled by quantum

interference even at the smallest distances.) What we will do is to extract from the field theory the short-distance singularity of the conductance between two strong contacts. Matching the result to the classically expected singularity will then determine  $f$ .

### 13.1 Classical point-contact conductance

The first thing to describe is the classical expectation. We will work in the infinite plane  $\mathbb{R}^2 \simeq \mathbb{C}$  with complex coordinate  $z = x_1 + ix_2$ . From linear response theory, the conductance,  $G$ , is a current-current correlation function,

$$G = \oint_{z \in C_1} \oint_{z' \in C_2} \langle j(z) j(z') \rangle , \quad (14)$$

where, for a system with interior contacts at points with coordinates  $z_1$  and  $z_2$ , the integration contours  $C_1$  and  $C_2$  are two disjoint cycles enclosing the points  $z_1$  resp.  $z_2$ . For a *classical* conductor, the current one-form  $j$  can be expressed in terms of a real boson field  $\varphi$  by

$$j = \frac{1}{i} \left( \frac{\partial \varphi}{\partial z} dz - \frac{\partial \varphi}{\partial \bar{z}} d\bar{z} \right) = \epsilon_{\mu\nu} \partial_\mu \varphi dx_\nu ,$$

and the expectation value  $\langle \bullet \rangle$  is defined by a Gaussian functional integral

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \bullet \exp \int d^2x \varphi \partial \bar{\partial} \varphi .$$

(We are working in scaled units here and will put back the proper units later.) As usual, the partition function  $Z$  normalizes the integral, so that  $\langle 1 \rangle = 1$ . The escape of probability flux through the contacts is modeled by imposing “absorbing” boundary conditions at these points:

$$\varphi(z_1) = \varphi(z_2) = 0 .$$

By the equation of motion for  $\varphi$  ( $\bar{\partial} \partial \varphi = 0$ ), the current is conserved:  $dj = 0$  on  $\mathbb{C} \setminus \{z_1 \cup z_2\}$ , and the expression (14) does not change under small deformations of  $C_1$  and  $C_2$ .

We now wish to know how the conductance  $G$  depends on  $|z_1 - z_2|$ , the distance between the two contacts. One way of computing the dependence is to use an analogy with classical 2d electrostatics, by reinterpreting  $\varphi$  as an electric potential,  $j$  as an electric flux *etc.* We can then view the functional integral average  $\langle \varphi(z) \varphi(z') \rangle$  as the Green function of a Poisson problem, which can be solved by the technique of conformal mapping. Alternatively we can compute  $G$  directly, by manipulation of the functional integral in the way detailed in the next subsection.

However, we will take the short cut of guessing the answer on dimensional grounds. Because the components of a conserved current have scaling dimension one, and a coordinate differential subtracts one dimension, the scaling dimension of the current one-forms  $j(z)$  integrated in (14) is zero. By conformal invariance of the free boson theory, the conductance  $G$  can therefore depend on  $|z_1 - z_2|$  only through the logarithm. On physical grounds,  $G$  must *decrease* with increasing distance between the contacts, so we expect  $G \sim (\ln |z_1 - z_2|)^{-1}$ . The constant of proportionality can be determined by comparison with the quasi-1d limit. According to Ohm's law, the conductance of a cylinder of circumference  $W$ , height  $|z_1 - z_2|$ , and conductivity  $\sigma$ , is  $G = \sigma W / |z_1 - z_2|$ . On replacing the (quasi-)1d Coulomb propagator  $|z_1 - z_2| / 2W$  by the 2d Coulomb propagator  $(2\pi)^{-1} \ln |z_1 - z_2|$ , we arrive at

$$G_{2d} = \frac{\pi\sigma}{\ln(|z_1 - z_2|/R)} , \quad (15)$$

which is the correct result. A length scale  $R$  was inserted in the argument of the logarithm to get the physical dimensions straight. (We may regard  $R$  as the size of the contacts, which sets the scale and serves as a short-distance cutoff.)

## 13.2 Classical conductance from field theory

Our next task is to reproduce the classical result (15) from the short-distance physics of the WZW model. As before, we consider an infinite system with interior point contacts placed at two positions  $z_1$  and  $z_2$ , and compute the conductance as a current-current correlation function.

To begin, recall from the end of Section 10 the parametrization  $g(x) = e^{Y(x)} e^{X(x)} g_0$ , where  $X(x)$  takes values in  $\text{Lie}(H)$ , and  $Y(x)$  in the complement  $\mathcal{P}$  of  $\text{Lie}(H)$  in  $\mathfrak{psl}(2|2)$ . The latter determines the field of Pruisken's model by  $Q = e^Y \Sigma_3 e^{-Y} = e^{2Y} \Sigma_3$ , while the former is post-Pruisken. From the supersymmetric nonlinear sigma model for disordered metals [9], the boundary condition for  $Q$  is known to be  $Q = \Sigma_3$  on any well-conducting boundary. By transcription to the present case, we require

$$Y(z_1) = Y(z_2) = 0 .$$

The linear field  $X$  is *not* subject to any such boundary condition. However, in the limit of vanishing regularization parameter  $\epsilon \rightarrow 0$ , we can exploit the global invariance of the field theory to set  $X = 0$  at *one* of the two contacts, say  $z_1$ . At that contact, we then have

$$g(z_1) = g_0 .$$

(We here ignore the subtle issue whether such a boundary condition is admissible in the presence of a Wess-Zumino term.)

By the use of current conservation, we may shrink the contours  $C_1$  and  $C_2$  to infinitesimal loops encircling the contacts  $z_1$  and  $z_2$ . In the vicinity of the first contact  $z_1$ , the WZW field  $g$  performs small fluctuations around the identity coset  $g_0$ . Therefore, if the second contact  $z_2$  is close enough to the first one, we may compute the current-current correlation function by expanding  $g$  in  $X$  and  $Y$ , and truncating at quadratic order. (Of course, the theory becomes nonlinear far from  $z_1$ , but the contributions from there cancel by supersymmetry and  $c = 0$ .) Thus the problem reduces to a free-field calculation.

From the variation of the action functional (8), we obtain the following expression for the conserved (left) current of the WZW model

$$J_\mu = \frac{1}{4\pi} (f^{-2} g^{-1} \partial_\mu g + ik\epsilon_{\mu\nu} g^{-1} \partial_\nu g) . \quad (16)$$

This current is  $\text{psl}(2|2)$ -valued. Linear response theory, in its transcription to the field-theoretic formalism for disordered electron systems, instructs us to expand the current in  $\text{psl}(2, 2)$  generators:  $J_\mu = \frac{1}{2} \sum_a J_\mu^a T^a$ , and pick any component  $J_\mu^a$  whose generator  $T^a$  *anticommutes* with the matrix  $\Sigma_3$ . Since the diagonal matrix  $\Sigma_3$  distinguishes between the advanced and retarded sector, such a component of the conserved current mixes retarded and advanced degrees of freedom, and its correlation function gives the conductance. To be definite, we take  $J_\mu^a$  to be a bosonic component of  $J_\mu$ , and the corresponding generator has the standard normalization  $\langle T^a, T^a \rangle = 2$ .

Because it suffices to do a free-field calculation, we may approximate

$$J_\mu^a \approx \frac{1}{8\pi} (f^{-2} \partial_\mu Y^a + ik\epsilon_{\mu\nu} \partial_\nu Y^a) ,$$

where  $Y^a$  is determined by  $Y = \frac{1}{2} \sum_a Y_a T^a$ . On integration along one of the cycles  $C_i$ , the first term yields the radial current flow emanating from or sinking into a contact, whereas the second term is topological and measures the vorticity of the flow around a contact. By the choice of boundary condition  $Y(z_1) = Y(z_2) = 0$ , the topological current is exact, and the vorticity must therefore vanish, so we may neglect the latter term. (Note that the topological current does have a finite effect in the presence of an insulating boundary [66].) Hence, if we simplify the notation by putting  $\varphi \equiv Y^a$ , we have  $J_\mu^a = (8\pi f^2)^{-1} \partial_\mu \varphi$ . To obtain the total current through a contour  $C_i$ , we need to integrate the (Hodge dual of the) current one-form,

$$j \equiv \epsilon_{\mu\nu} J_\mu^a dx_\nu = (8\pi f^2)^{-1} \epsilon_{\mu\nu} \partial_\mu \varphi dx_\nu .$$

With this identification, the conductance is again given by (14), the functional integral now being

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \bullet \exp \frac{1}{4\pi f^2} \int d^2x \varphi \partial \bar{\partial} \varphi .$$

The normalizing denominator arises from integrating over the components of  $Y$  other than  $Y^a$  (while integration over  $X$  cancels the determinant from the auxiliary complex boson). Hence we are back to the problem considered in the previous subsection. Instead of guessing the answer for  $G$ , let us now compute the conductance explicitly, carrying along the necessary scale factors.

The first step is to make the boundary conditions explicit in the functional integral. For that we place Dirac  $\delta$ -distributions on the field  $\varphi$  at  $z_1$  and  $z_2$ . The “partition function”  $Z$  then takes the form

$$Z = \int \mathcal{D}\varphi \delta[\varphi(z_1)] \delta[\varphi(z_2)] \exp \frac{1}{4\pi f^2} \int d^2x \varphi \partial \bar{\partial} \varphi ,$$

where  $\delta[\varphi] = \lim_{\varepsilon \rightarrow 0} (\pi\varepsilon)^{-1/2} e^{-\varphi^2/\varepsilon}$ . A similar formula holds for the numerator of the current-current correlation function.

The next step is to “fuse” the  $\delta$ -distribution  $\delta[\varphi(z_1)]$  with the operator  $O_1[\varphi] \equiv \oint_{z \in C_1} j(z)$ , by contracting  $C_1$  to the point  $z_1$  (and similarly with  $C_2$  at  $z_2$ ) to produce a *local* operator. The dominant term of the operator product expansion is extracted as follows. We put  $z - z_1 = e^{\tau + i\sigma}$  ( $\tau, \sigma$  are the coordinates that are used for what is called “radial quantization” around the point  $z_1$ ), and take  $C_1$  to be a circle  $C_1 = \{z \in \mathbb{C} : |z - z_1| = \varepsilon\}$ . Then

$$O_1[\varphi] = \frac{1}{8\pi f^2} \oint_{\tau=\ln \varepsilon} \frac{\partial \varphi}{\partial \tau} d\sigma .$$

To go further we use a simple analogy. Consider a quantum particle in one dimension with mass  $m$  and coordinate  $x$ . Then we know that the velocity  $\dot{x} = dx/d\tau$  in Feynman’s imaginary-time path integral translates into the operator  $m^{-1}d/dx$  in Schrödinger quantum mechanics. Moreover, for a free particle,

$$\int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x \, e^{-(m/2) \int_0^T \dot{x}^2 d\tau} \dot{x}(\tau_0) = \left\langle x_f \left| \exp \left( \frac{T}{2m} \frac{d^2}{dx^2} \right) m^{-1} d/dx \right| x_i \right\rangle ,$$

independent of the time  $\tau_0$  where the velocity is evaluated. Similarly, if  $\varphi_i(\sigma)$  and  $\varphi_f(\sigma)$  are prescribed functions  $S^1 \rightarrow \mathbb{R}$ , then in the present problem we have

$$\begin{aligned} & \int_{\varphi(\bullet, T_i)=\varphi_i}^{\varphi(\bullet, T_f)=\varphi_f} \mathcal{D}\varphi \left( \oint_{\tau=\ln \varepsilon} \frac{\partial \varphi}{\partial \tau} d\sigma \right) \exp \frac{1}{4\pi f^2} \int_A d^2x \varphi \partial \bar{\partial} \varphi \\ &= 8\pi f^2 \oint d\sigma \left\langle \varphi_f \left| e^{-(T_f - T_i)\mathcal{H}} \frac{\delta}{\delta \varphi(\sigma)} \right| \varphi_i \right\rangle . \end{aligned}$$

Here  $A$  denotes the annulus  $T_i < \tau < T_f$ ,  $0 \leq \sigma < 2\pi$ , and  $\mathcal{H}$  is the Hamiltonian of the radially quantized theory. Thus, by shrinking  $C_1$  to the point  $z_1$  we obtain

$$O_1[\varphi] \delta[\varphi(z_1)] \longrightarrow \frac{\partial}{\partial \varphi(z_1)} \delta[\varphi(z_1)] \equiv \delta'[\varphi(z_1)] .$$



Doing the same at the other contact and returning from the radially quantized theory to the functional integral, we arrive at the expression

$$G = \frac{1}{Z} \int \mathcal{D}\varphi \, \delta'[\varphi(z_1)] \, \delta'[\varphi(z_2)] \, \exp \frac{1}{4\pi f^2} \int d^2x \, \varphi \partial \bar{\partial} \varphi .$$

This is easy to compute by Fourier expanding the  $\delta$ -distributions in terms of “vertex operators”:

$$\delta[\varphi] = \int_{\mathbb{R}} \frac{d\lambda}{2\pi} \exp i\lambda\varphi .$$

In the case of the partition function  $Z$ , a standard result for Gaussian functional integrals gives

$$\begin{aligned} Z &= \int_{\mathbb{R}} \frac{d\lambda}{2\pi} \int_{\mathbb{R}} \frac{d\lambda'}{2\pi} \int \mathcal{D}\varphi \, \exp \left( (4\pi f^2)^{-1} \int d^2x \, \varphi \partial \bar{\partial} \varphi + i\lambda\varphi(z_1) + i\lambda'\varphi(z_2) \right) \\ &= \text{const} \times \int_{\mathbb{R}} \frac{d\lambda}{2\pi} \exp -2\pi f^2 \lambda^2 K(z_1, z_2) , \end{aligned}$$

where  $K$  is the Coulomb propagator,  $K(z_1, z_2) = (\partial\bar{\partial})^{-1}(z_1, z_2)$ . Doing the same computation for the numerator of the current-current correlation function and taking the ratio, we find

$$G = (4\pi f^2 K(z_1, z_2))^{-1} = \frac{1/8f^2}{\ln(|z_1 - z_2|/R)} .$$

From the previous subsection, we know that the numerator of the last expression equals  $\pi\sigma_{xx}$ , with  $\sigma_{xx}$  the classical conductivity. Hence, we have the result

$$f^2 = (8\pi\sigma_{xx})^{-1} . \tag{17}$$

If we appeal to the semicircle relation (13), which gives  $\sigma_{xx} = 1/2$  for the critical value  $\sigma_{xy} = 1/2$ , we get  $f^2 = 1/4\pi$ . Note that for this value of the coupling the current (16) is not holomorphic, and the field  $g$  does not separate into left-moving and right-moving waves. This will make it harder to obtain analytical solutions than in the conventional (Kac-Moody) case.

## 14 Quantum point-contact conductance

With both couplings of the WZW model now determined, we put the theory to test on a nontrivial transport coefficient: the conductance between two interior contacts in the *quantum limit* of point-size contacts. This observable was the object of a recent study [26] where heavy analytical machinery was combined with numerical simulation. Starting from the Chalker-Coddington model with

two point contacts separated by a distance  $r$ , the following expression for the  $q$ -th moment of the point-contact conductance was derived:

$$\langle G^q \rangle = \int_0^\infty d\mu(\lambda) C_q(\lambda) r^{-2\Delta_\lambda} , \quad (18)$$

where

$$d\mu(\lambda) = \frac{\lambda}{2} \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda , \quad C_q(\lambda) = \left| \frac{\Gamma\left(q - \frac{1+i\lambda}{2}\right)}{\Gamma(q)} \right|^2 .$$

(Note that the conductance of a phase-coherent quantum system *fluctuates* as a function of disorder [67, 68], and for a complete description we need to compute the entire distribution function or, equivalently, all of the moments of the conductance. Note also that the use of the symbol  $\lambda$  is not accidental: this parameter plays the same role here as in the classical calculation of subsection 13.1.) The factors  $d\mu(\lambda)$  and  $C_q(\lambda)$  are purely kinematical, the first one being a Plancherel measure and the second the square of a Clebsch-Gordan coefficient, both of which are determined by representation theory and harmonic analysis. All dynamical information from the field theory resides in the  $r$ -dependent factor  $r^{-2\Delta_\lambda}$ . The exponents  $\Delta_\lambda$  are the scaling dimensions of certain scaling fields  $\phi_\lambda$ . To elucidate their origin, we now give a rough summary of how the formula for  $\langle G^q \rangle$  is obtained (the details are found in [26]).

The first step is to cast the Chalker-Coddington network model in the form of a supersymmetric vertex model [35] with global  $\text{GL}(2|2)$  symmetry, by using the color-flavor transformation in the way indicated in Section 2. The partition function of the vertex model is a sum over (classical) degrees of freedom situated on the links of the network. They take values in the  $\text{GL}(2|2)$  modules  $V$  and  $V^*$  that were defined in Section 4. In the vertex-model representation, the point-contact conductance assumes the form of a two-point correlation function,

$$\langle G^q \rangle = \langle O_q^+(0) O_q^-(r) \rangle ,$$

where  $O_q^\pm$  are operators that are viewed as elements of  $V \otimes V^*$ , and are given in terms of the vacuum  $|0\rangle$  and the superspin generators  $S_{ij}$  of Section 4 by

$$O_q^+ = (S_{02})^q |0\rangle \otimes |\bar{0}\rangle , \quad O_q^- = |0\rangle \otimes (S_{20})^q |\bar{0}\rangle .$$

Precisely speaking, to make sense of the expression for  $\langle G^q \rangle$ , one canonically identifies  $V \otimes V^*$  with  $\text{End}(V)$ , and computes the vertex model sum with  $O_q^\pm \in \text{End}(V)$  acting on the superspin at the corresponding link.

The second step is to Fourier-analyze the elements  $O_q^\pm$  of the tensor product  $V \otimes V^*$ . Because the vertex model has global  $\text{SL}(2|2)$  symmetry, one wants to decompose  $V \otimes V^*$  into irreducibles w.r.t. this group or, with no loss, w.r.t.  $\text{PSL}(2|2)$ . This decomposition was first described in Section 5.2 of [45].

It turns out that the only representations that appear are those of a *continuous* series, labelled by  $\lambda \in \mathbb{R}^+$ , which is essentially the same as the principal continuous series of unitary representations of  $\text{SU}(1, 1)$  — the textbook example for harmonic analysis on a noncompact group. In a self-explanatory notation, the Fourier decomposition of the operators  $O_q^\pm$  is written

$$O_q^+ = \int_0^\infty d\mu(\lambda) \langle Vq, V^*0|\lambda q \rangle \phi_{\lambda q}, \quad O_q^- = \overline{O_q^+}. \quad (19)$$

These decompositions are inserted into the correlation function  $\langle O_q^+(0) O_q^-(r) \rangle$ . Adopting the phase convention  $\overline{\phi_{\lambda q}} = \phi_{\lambda - q}$ , denoting the product of Clebsch-Gordan coefficients by

$$C_q(\lambda) = \left| \langle Vq, V^*0|\lambda q \rangle \right|^2,$$

and postulating

$$\langle \phi_{\lambda q}(0) \phi_{\lambda' - q}(r) \rangle = \frac{\delta(\lambda - \lambda')}{m(\lambda)} r^{-2\Delta_\lambda}, \quad (20)$$

where  $m(\lambda)$  is defined by  $d\mu(\lambda) = m(\lambda)d\lambda$ , we then arrive at (18).

Let us pause here for a moment to insert the following remark. In [29] the representation theory of  $\text{PSL}(2|2)$  was discussed in some detail, with the focus being on *highest-weight* representations. The existence of representations without highest weight was mentioned, but only parenthetically, as the authors of [29] “... do not have the need for those”. In our case, the situation is quite the opposite! Highest-weight representations (or, for that matter, lowest-weight representations) play no role for the conductance, and all the representations labelled by  $\lambda$  are of another type. This feature is forced by the fact that the representations of  $\text{PSL}(2|2)$  on the modules  $V$  and  $V^*$  are *unitary* w.r.t. to a subgroup  $\text{SU}(1, 1) \times \text{SU}(2)$ , so the decomposition of  $V \otimes V^*$  into  $\text{PSL}(2|2)$  irreducibles is exhausted by unitary representations of this subgroup. It is well known that the unitary representations of the noncompact group  $\text{SU}(1, 1)$  organize into discrete and continuous series. All representations that occur in the tensor product  $V \otimes V^*$  contain the vector  $|0\rangle \otimes |\bar{0}\rangle \in V \otimes V^*$ , which is stable under the action of the compact  $\text{U}(1)$  subgroup of  $\text{SU}(1, 1)$ . Thus, as far as  $\text{SU}(1, 1)$  is concerned, our problem amounts to doing harmonic analysis on  $\text{SU}(1, 1)/\text{U}(1) \simeq \mathbb{H}^2$ . This eliminates the discrete unitary series of  $\text{SU}(1, 1)$  representations (all of which are given by holomorphic sections of associated line bundles  $\text{SU}(1, 1) \times_{\text{U}(1)} \mathbb{C}_\mu$  with *nonzero*  $\text{U}(1)$  charge  $\mu$ ), leaving only the continuous series. The members of the latter are non-algebraic and do not contain any highest-weight or lowest-weight vector.

Next, we indicate how to construct the functions  $\phi_{\lambda q}$  appearing in the Fourier decomposition of  $O_q^+ \in V \otimes V^*$ . For that we adopt the notation  $G = \text{PSU}(1, 1|2)$  and  $K = \text{PS}(\text{U}(1|1) \times \text{U}(1|1))$ . By a supersymmetric version of the Borel-Weil correspondence, the module  $V$  can be viewed as the space of holomorphic

sections of a line bundle  $G \times_K \mathbb{C}_m$ , while  $V^*$  is the space of antiholomorphic sections of the conjugate bundle  $G \times_K \mathbb{C}_{m^*}$ . (In fact, this is precisely how  $V$  and  $V^*$  first arose in [45].  $m$  is a one-dimensional representation of  $K$  by the superdeterminant, and  $m^* = m^{-1}$ .) From this viewpoint, the tensor product  $V \otimes V^*$  lies inside some space of square integrable functions on  $G/K$ , and the Fourier decomposition of  $V \otimes V^*$  becomes a sub-problem of the problem of harmonic analysis on  $G/K$ . (This is familiar from quantum mechanics: the modules  $V$  and  $V^*$  are Hilbert spaces that arise by geometric quantization of the classical phase space  $G/K$ , and by multiplying the “wave functions” of  $V$  with the conjugate wave functions of  $V^*$ , we recover  $G/K$  in the form of “Wigner functions” with a fuzzy resolution. Note also the following. To decompose the tensor product  $V \otimes V^*$ , we need to diagonalize the low-order Casimir invariants of  $G$ . In the present case, consideration of the quadratic Casimir turns out to be sufficient. Under the embedding of  $V \otimes V^*$  into a function space on  $G/K$ , the quadratic Casimir corresponds to a second-order differential operator invariant w.r.t.  $G$ , the Laplace-Beltrami operator.) The latter problem was solved by a supersymmetric adaptation of Harish-Chandra theory in [69, 70, 71]. We now give a quick taste of the basic idea.

Let  $G_{\mathbb{C}} = N_{\mathbb{C}} A_{\mathbb{C}} K_{\mathbb{C}}$  be an Iwasawa decomposition [72] of  $G_{\mathbb{C}} \equiv \mathrm{PSL}(2|2)$ . (Compact groups such as  $\mathrm{SU}(2) \subset G$  do not admit an Iwasawa decomposition, so we are forced to work on the complexification  $G_{\mathbb{C}}$ .) The “radial” factor  $A_{\mathbb{C}}$  is a maximal abelian subgroup with Lie algebra contained in  $\mathcal{P}$  (where  $\mathcal{P}$  is defined by  $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(K_{\mathbb{C}}) + \mathcal{P}$ ). The Iwasawa decomposition determines a radial function  $A : G_{\mathbb{C}} \rightarrow \mathrm{Lie}(A_{\mathbb{C}})$  by

$$g = n(g) e^{A(g)} k(g) .$$

This function is left-invariant under  $N_{\mathbb{C}}$ , right-invariant under  $K_{\mathbb{C}}$ , and restricts to a (complex-valued) function on  $\mathbf{X}_{A_1|A_1}$ . Because  $A_{\mathbb{C}}$  normalizes  $N_{\mathbb{C}}$ , the radial part  $L_A$  of the Laplace-Beltrami operator  $L$  on  $G$  is invariant under left translations by elements of  $A_{\mathbb{C}}$ . As a result,  $L_A$  is a differential operator with *constant coefficients*, and its eigenfunctions are simple exponentials. Thus, any weight  $\mu : \mathrm{Lie}(A_{\mathbb{C}}) \rightarrow \mathbb{C}$  gives rise to an eigenfunction  $\Phi_{\mu}(g) = \exp \mu(A(g))$  of  $L_A$ , and hence of  $L$ . (In general, such a function will be well-defined only locally, and global consistency in the compact sector imposes an integrality condition on  $\mu$ .) With  $\Phi_{\mu}(g)$ , every rotated function  $\Phi_{\mu}(kg)$  ( $k \in G$ ) also is an eigenfunction of  $L$ . The question addressed by harmonic analysis is which of the  $\mu$  to use in constructing the Fourier transform and its inverse, and what is the measure in Fourier space (the Plancherel measure).

For the case at hand, the answer is known from [71]. Since  $\dim(A_{\mathbb{C}}) = 2$ , the weights are parametrized by two quantum numbers: a continuous number  $\lambda \in \mathbb{R}^+$  reflecting the  $\mathrm{SU}(1,1)$  content, and a discrete number  $l \in 2\mathbb{N} - 1$  labelling representations of  $\mathrm{SU}(2)$  with integer spin. On restriction from the

complete function space on  $G/K$  to the subspace  $V \otimes V^*$ , the second quantum number gets frozen at the minimal value  $l = 1$  (this apparently is related to the level of the WZW model being minimal), and we are left with a single parameter  $\lambda \in \mathbb{R}^+$ . The corresponding eigenfunctions  $\Phi_\lambda$  of the Laplace-Beltrami operator  $L$  can be constructed quite explicitly. In the parametrization of  $G/K$  by the complex supermatrices  $Z, \tilde{Z}$  introduced in (2), they are

$$\Phi_\lambda = \text{SDet} \left( \frac{(1 - \tilde{Z})(1 - Z)}{(1 - \tilde{Z}Z)} \right)^{(1+i\lambda)/2}.$$

These are eigenfunctions of  $L$  with eigenvalue  $-(\lambda^2 + 1)$ . Other eigenfunctions with the same eigenvalue can be constructed by acting with the symmetry group  $G$  on  $\Phi_\lambda$ . The set of functions so obtained form an “eigenfunction” representation space,  $V_\lambda$ , of  $G$ . We will not go into the details of its construction here. Suffice it to say that the function  $\phi_{\lambda q}$  appearing in the Fourier decomposition (19) is the element of  $V_\lambda$  with the same weights w.r.t. the diagonal generators  $S_{ii} \otimes 1 + 1 \otimes S_{ii}$  of  $G$  as  $O_q^+ \in V \otimes V^*$ .

The  $\Phi_{\lambda q}$  extend to functions on  $\mathbf{X}_{A_1|A_1}$  in the natural way. We can therefore consider the two-point correlator  $\langle \phi_{\lambda q}(0) \phi_{\lambda' - q}(r) \rangle$  in the  $A_1|A_1$  WZW model. On general grounds, such a correlation function is of the form (20) for a conformal field theory. Taking that formula for granted, the remaining problem is to compute the scaling dimension  $\Delta_\lambda$  of  $\phi_{\lambda q}$ . By  $G$ -invariance of the field theory,  $\Delta_\lambda$  is the same for every vector of the representation space  $V_\lambda$ . In particular,  $\Delta_\lambda$  agrees with the scaling dimension of the function  $\Phi_\lambda$  generating the eigenfunction representation. Leaving aside the possibility of an exact determination, we can compute  $\Delta_\lambda$  from the algebraic decay of the correlator  $\langle \Phi_\lambda(0) \overline{\Phi_{\lambda'}(r)} \rangle$ . Since  $\Phi_\lambda$  is simply an exponential in Iwasawa coordinates, perturbation theory is easy to implement and gives

$$\Delta_\lambda = f^2(\lambda^2 + 1) \tag{21}$$

to leading order. In Ref. [29] this result (for highest-weight representations, but it doesn’t make any difference) was conjectured to be *exact*. I have no nonperturbative proof of that conjecture but also no evidence against it. (As we are about to see, the numerics of [26] supports the conjecture.) We shall therefore *assume* (21) to be true, leaving its verification as an open problem for future work.

Given the formulas (18) and (21), we have a complete description of the quantum point-contact conductance. For the purpose of numerical computer simulation, the best observable to consider is the *typical* conductance,  $\exp\langle G \rangle$ . This statistic is determined by typical events (unlike the average point-contact conductance, which is dominated by rare events), and can therefore be computed with high statistical accuracy. By analytically continuing (18) from the positive

integers  $q$  to the vicinity of  $q = 0$ , and using  $\langle G^q \rangle = 1 + q\langle \ln G \rangle + \mathcal{O}(q^2)$ , one obtains

$$\langle \ln G \rangle = -2i \frac{d}{d\lambda} r^{-2\Delta_\lambda} \big|_{\lambda=i} = -8f^2 \ln r$$

or upon exponentiation,

$$\exp \langle \ln G \rangle = r^{-X_t}, \quad X_t = 8f^2.$$

Thus the typical point-contact conductance decays with distance as a pure power. (Note from (18) that this is a unique feature not shared by the average, the variance, or any higher moment.) In Ref. [26], numerical data for  $\langle \ln G \rangle$  collected from the Chalker-Coddington network model, were plotted as a function of  $\ln r$ . The data nicely fell on a straight line, and the best fit (properly taking into account the statistical errors) was obtained with  $X_t = 0.640 \pm 0.009$ . This agrees with the analytical prediction  $X_t = 2/\pi \approx 0.637$ , following from  $X_t = 8f^2$  and  $f^2 = (8\pi\sigma_{xx})^{-1}$  together with the value  $\sigma_{xx} = 1/2$  for the classical conductivity of the network model.

## 15 Discussion

If the present proposal is correct – I believe I have given substantial evidence in its favor and submit it now to the community for judgment –, opens a new chapter in the theory of the plateau transition of the integer quantum Hall effect. For the first time, we have related the transition to the Lagrangian of a field theory that enjoys manifest conformal invariance, namely the  $A_1|A_1$  WZW model. This link finally opens the possibility of administering some nonperturbative tools of conformal field theory. An intriguing aspect of our proposal is that essentially the same theory appeared in recent work on string propagation in  $\text{AdS}_3$  backgrounds. Hopefully, this coincidence will spur communication between fields and enhance the unity of theoretical physics. It also corroborates the status of the quantum Hall effect as one of the most exciting and profound phenomena brought forth by recent condensed matter physics.

A remarkable consequence of our proposal is that critical properties in the QH class can be universal only in a restricted sense, as the true marginality of the coupling  $f$  implies the existence of a line of fixed points, not just an isolated fixed point. A priori, one expects that the critical RG trajectories for different members of the QH class (the network model, for example, the Gaussian white noise random potential projected on the lowest Landau level, or disordered Dirac fermions) terminate on different points of the fixed line, leading to a variable set of exponents. This seems to be at variance, at first sight, with what has been observed in numerical and real experiments. However, as we have argued, the marginal coupling  $f$  is completely determined by the (renormalized) conductivity governing the classical motion of electrons near an absorbing boundary.

Universality will therefore persist as long as the classical diffusion over short distances is governed by a universal conductivity. It is encouraging that Ruzin and Feng have argued the latter to be true on the basis of a two-fluid model of incoherent QH systems.

It has been my contention for a number of years that, from a theoretical perspective, the most natural probe of critical transport at the plateau transition is the conductance between two *interior point contacts*. Such a conductance is represented by a two-point function of primaries in the conformal field theory. Moreover, the operators appearing in the field-theoretic representation of a point contact are operators that create *normalizable* states lying inside the physical Hilbert space. (In contrast, the operator for, say, the local density of states creates unnormalizable states lying outside the physical space.) The point-contact conductance therefore is the basic object of the theory, and any theoretical development is well advised to start from there. In the present work, we used the short-distance singularity (the classical limit) of the conductance between two strong contacts to determine the marginal coupling  $f$ . By matching onto the classical conductivity  $\sigma_{xx} = 1/2$  of the Chalker-Coddington network model at criticality, we found an *irrational* coupling  $f^2 = 1/4\pi$ . We then explained why this value predicts for the typical point-contact conductance of the network model an exponent  $X_t = 2/\pi$ , which is confirmed by recent numerical results.

The present work leaves a number of open questions, but provides a systematic framework for answering them. The most urgent need is a calculation of the exponent for the localization length,  $\nu$ . The answer for  $\nu$  is not immediate from our results, as this requires the renormalization of an operator with two derivatives. A great many theoretical proposals for this number have been made in the literature, the most widely cited one being  $\nu = 7/3$  [73], and the most recent  $\nu = 20/9$  [74]. From the present theory it seems likely that  $\nu$  will be none of these, but some irrational number.

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## References

- [1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45** (1980) 494
- [2] D.C. Tsui, H.L. Stoermer, and A.C. Gossard, Phys. Rev. Lett. **48** (1982) 1559

- [3] R.E. Prange and S.M. Girvin, eds., *The Quantum Hall Effect* (Springer-Verlag, Berlin, 1987)
- [4] B. Huckestein, Rev. Mod. Phys. **67** (1995) 357
- [5] H.P. Wei, D.C. Tsui, M.A. Palaanen, and A.M.M. Pruisken, Phys. Rev. Lett. **61** (1988) 1294
- [6] A.M.M. Pruisken, Phys. Rev. Lett. **61** (1988) 1297
- [7] A.M.M. Pruisken, Nucl. Phys. B **235** (1984) 277
- [8] H.A. Weidenmüller, Nucl. Phys. B **290** (1987) 87
- [9] K.B. Efetov, Adv. Phys. **32** (1983) 53
- [10] D.E. Khmel'nitskii, JETP Lett. **38** (1983) 454
- [11] H. Levine, S.B. Libby, and A.M.M. Pruisken, Nucl. Phys. B **240** (1984) 30, 49, 71
- [12] I. Affleck, Nucl. Phys. B **265** (1986) 409
- [13] F.D.M. Haldane, Phys. Rev. Lett. **50** (1983) 1153
- [14] I. Affleck and F.D.M. Haldane, Phys. Rev. B **36** (1987) 5291
- [15] J.J.M. Verbaarschot and M.R. Zirnbauer, J. Phys. A **18** (1985) 1093
- [16] M.R. Zirnbauer, "Another critique of the replica trick", cond-mat/9903338
- [17] I. Affleck, Nucl. Phys. B **305** (1988) 582
- [18] K. Gawedzki and A. Kupiainen, Nucl. Phys. B **320** (1989) 625
- [19] M.R. Zirnbauer, J. Math. Phys. **37** (1996) 4986
- [20] J. Bernstein, lectures on supersymmetry, given at the IAS semester on quantum field theory (1996)
- [21] L. Rozansky and H. Saleur, Nucl. Phys. B **376** (1992) 461
- [22] A.W.W. Ludwig, M.P.A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B **50** (1994) 7526
- [23] D. Bernard, "(Perturbed) conformal field theory applied to 2d disordered systems: an introduction", hep-th/9509137
- [24] E. Witten, Commun. Math. Phys. **92** (1984) 455



- [25] I. Affleck, Phys. Rev. Lett. **55** (1985) 1355
- [26] M. Janssen, M. Metzler, and M.R. Zirnbauer, “Point-contact conductances at the quantum Hall transition”, cond-mat/9810319
- [27] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B **247** (1984) 83
- [28] N. Berkovits, C. Vafa, and E. Witten, “Conformal field theory of AdS background with Ramond-Ramond flux”, hep-th/9902098
- [29] M. Bershadsky, S. Zhukov, and A. Vaintrob, “PSL( $n|n$ ) sigma model as a conformal field theory”, hep-th/9902180
- [30] J.T. Chalker and P.D. Coddington, J. Phys. C **21** (1987) 2665
- [31] R. Klesse and M. Metzler, Europhys. Lett. **32** (1995) 229
- [32] R. Klesse and M. Metzler, Phys. Rev. Lett. **79** (1997) 721
- [33] S. Kettemann and A. Tsvelik, “Information about the integer quantum Hall transition extracted from the autocorrelation function of spectral determinants”, cond-mat/9809012
- [34] M.R. Zirnbauer, “The color-flavor transformation and a new approach to quantum chaotic maps”, chao-dyn/9810016
- [35] M.R. Zirnbauer, J. Math. Phys. **38** (1997) 2007; *ibid.* **40** (1999) 2197(E)
- [36] M.R. Zirnbauer, J. Phys. A **28** (1996) 7113
- [37] R.J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, London, 1982)
- [38] F. Wegner, Z. Phys. B **35** (1979) 207
- [39] L. Schäfer and F. Wegner, Z. Phys. B **38** (1980) 113
- [40] K.B. Efetov, A.I. Larkin, and D.E. Khmel’nitskii, Sov. Phys. JETP **52** (1980) 568
- [41] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B **379** (1992) 602
- [42] R. Shankar and N. Read, Nucl. Phys. B **336** (1990) 457
- [43] N. Read, private communication (Aspen, 1993)
- [44] R. Howe, Trans. Amer. Math. Soc. **313** (1989) 539

- [45] M.R. Zirnbauer, *Annalen der Physik* **3** (1994) 513
- [46] D.-H. Lee, *Phys. Rev. B* **50** (1994) 10788
- [47] J. Kondev and J.B. Marston, *Nucl. Phys. B* **497** (1997) 639
- [48] J.B. Marston and S.-W. Tsai, “The Chalker-Coddington network model is quantum critical”, *cond-mat/9812261*.
- [49] R. Gade, *J. Phys. A* **31** (1998) 4909
- [50] M. Stone, *Nucl. Phys. B* **327** (1989) 399
- [51] F. Wegner, *Z. Phys. B* **44** (1981) 9
- [52] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Nucl. Phys. B* **371** (1992) 269
- [53] R. Gade and F. Wegner, *Nucl. Phys. B* **360** (1991) 213
- [54] R. Gade, *Nucl. Phys. B* **398** (1993) 499
- [55] A.V. Andreev, B.D. Simons, and N. Taniguchi, *Nucl. Phys. B* **432** (1994) 487
- [56] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York, 1978)
- [57] F.A. Berezin, *Introduction to Superanalysis* (Reidel, Dordrecht, 1987)
- [58] A. Schwarz and O. Zaboronsky, *Comm. Math. Phys.* **183** (1997) 463
- [59] V.G. Kac, *Adv. Math.* **26** (1979) 8
- [60] R. Bundschuh and M.R. Zirnbauer, still in preparation
- [61] S.P. Novikov, *Uspekhi Mat. Nauk* **37:5** (1982) 3
- [62] E. Witten, *Nucl. Phys. B* **371** (1992) 191
- [63] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, *Nucl. Phys. B* **279** (1984) 333
- [64] V. Gurarie, “*c*-theorem for disordered systems”, *cond-mat/9808063*.
- [65] I.M. Ruzin and S. Feng, *Phys. Rev. Lett.* **74** (1995) 154
- [66] S. Xiong, N. Read, and A.D. Stone, *Phys. Rev. B* **56** (1997) 3982
- [67] M. Janssen, *Phys. Rep.* **295** (1998) 1

- [68] D.H. Cobden and E. Kogan, Phys. Rev. B **54** (1996) 17316
- [69] M.R. Zirnbauer, Commun. Math. Phys. **141** (1991) 503
- [70] M.R. Zirnbauer, Phys. Rev. Lett. **69** (1992) 1584
- [71] A.D. Mirlin, A. Müller-Groeling, and M.R. Zirnbauer, Ann. Phys. **236** (1994) 325
- [72] S. Helgason, *Groups and geometric analysis* (Academic Press, New York, 1984)
- [73] G.V. Mil'nikov and I.M. Sokolov, JETP Lett. **48** (1988) 536
- [74] A. LeClair, "Gauge invariance and the critical properties of quantum Hall plateau transitions", cond-mat/9904414